

An action principle for Vasiliev's four-dimensional higher-spin gravity

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ABSTRACT. We provide Vasiliev's fully nonlinear equations of motion for bosonic gauge fields in four spacetime dimensions with an action principle. We first extend Vasiliev's original system with differential forms in degrees higher than one. We then derive the resulting duality-extended equations of motion from a variational principle based on a generalized Hamiltonian sigma-model action. The generalized Hamiltonian contains two types of interaction freedoms: One set of functions that appears in the Q-structure of the generalized curvatures of the odd forms in the duality-extended system; and another set depending on the Lagrange multipliers, encoding a generalized Poisson structure, *i.e.* a set of polyvector fields of ranks two or higher in target space. We find that at least one of the two sets of interaction-freedom functions must be linear in order to ensure gauge invariance. We discuss consistent truncations to the minimal Type A and B models (with only even spins), spectral flows on-shell and provide boundary conditions on fields and gauge parameters that are compatible with the variational principle and that make the duality-extended system equivalent, on shell, to Vasiliev's original system.

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1 Introduction

The natural geometric setting for higher-spin gauge theory is *unfolded dynamics* [1, 2, 3, 4]. This formalism, when applied to field theories with local propagating degrees of freedom, uses *exterior differential systems* with infinitely many zero-forms whose integration constants represent all the local information of the on-shell curvatures, usually referred to as the Weyl tensors. An exterior differential system (see e.g. [5, 6] and refs. therein) is an ideal I in the graded ring of locally defined differential forms on a smooth manifold M that is closed under the operation of exterior differentiation. An integral manifold of a differential system is an immersed submanifold of M on which each form in I restricts to zero. In the unfolded language, the latter forms are called *generalized curvatures*, and the integral manifold becomes a classical solution.

Considering retrospectively the works [7, 8, 9, 10, 11], one sees that these formulations of supergravities are examples of unfolded systems, *i.e.* exterior differential systems with infinite towers of Weyl zero-forms, though the locality of supergravity implies that all the dynamic content can be accessed (in the metric phase) by only considering the constraints on the forms in strictly positive degrees, thereby explaining why the authors of [10, 9, 8] did not consider the constraints on the generalized one-form curvatures for the Weyl tensors.

There exists a canonical framework for seeking off-shell formulations of unfolded dynamics based on generalized Poisson sigma models [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22], and [23, 24, 25, 26, 27, 28]. Although originally applied to topological systems, these models can be adapted relatively straightforwardly to unfolded systems, with their characteristic quasi-topological infinite towers of zero-forms, leading to the key physical problem as to whether this class of unfolded deformation quantum field theories actually contains standard relativistic quantum fields; see also [29, 30, 31] for recent developments.

In this paper we shall address this issue by using the fully non-linear and background-independent Vasiliev equations in four spacetime dimensions [2, 32, 33]. These equations possess an algebraic structure that enables us to construct a generalized Hamiltonian action with nontrivial QP -structures, and have geometric structures which allows to construct additional boundary deformations. In this paper we focus on the bulk part of the Hamiltonian action, leaving various deformations on submanifolds to future works. In fact, already in [3], such an action principle was proposed, which however did not contain any P -structure.

We wish to stress that, unlike the original Fronsdal programme, which attempts to formulate higher-

spin gauge theory off shell in a perturbative expansion around constantly curved spacetime, the work in this paper provides a background-independent formulation in terms of master fields living in the correspondence space, *i.e.* the local product of a non-commutative phase-spacetime containing the commutative spacetime as a Lagrangian submanifold and a non-commutative twistor space. Strictly speaking, the Vasiliev system has a huge classical solution space that admits many different perturbative expansions of which only some reduce to Fronsdal systems (with cosmological constant).

2 Duality extension on shell

2.1 Duality extended bosonic models

Our starting point is Vasiliev's on-shell formulation of higher-spin gravity in four spacetime dimensions [2, 32, 33] based on combining free differential algebra and the twistor map (see Appendix D).

Vasiliev's equations of motion provide a particular example of formulation of a classical field theory using free differential algebras, sometimes referred to as unfolded dynamics. In general, unfolded systems can be extended by adding forms in higher degrees. In particular, if the underlying differential algebra contains central and closed elements in degrees $\{0, 2, 4, \dots\}$, also the structure constants can be extended from the real numbers (in degree zero) to general central elements. If this extension is nontrivial, that is, if it cannot be removed by a field redefinition, then we refer to the resulting extended system as a duality extension of the original system. The duality-extended system contains the original system as a consistent subsystem, and this subsystem sources the duality-extended sector via nontrivial couplings involving central elements of positive degrees (see Appendix B for a more detailed discussion).

Vasiliev's equations can be extended adding forms in higher degrees as follows:

$$A = \sum_{p=1,3,\dots} A_{[p]}, \quad B = \sum_{p=0,2,\dots} B_{[p]}, \quad (2.1)$$

where $A_{[p]}$ and $B_{[p]}$ are locally defined differential forms of total degree p belonging to the algebra of bosonic forms with generic elements

$$f = \sum_{p=0}^{\infty} f_{[p]}(X^M, dX^M; Z^\alpha, dZ^\alpha; Y^\alpha; k, \bar{k}), \quad (2.2)$$

$$f_{[p]}(\lambda dX^M; \lambda dZ^\alpha) = \lambda^p f_{[p]}(dX^M; dZ^\alpha), \quad (2.3)$$

for complex parameters λ (we suppress the irrelevant variables whenever ambiguities cannot arise), where X^M are commuting coordinates, $(Y^\alpha, Z^\alpha) = (y^\alpha, \bar{y}^{\dot{\alpha}}; z^\alpha, \bar{z}^{\dot{\alpha}})$ are non-commutative twistor-space coordinates and k and \bar{k} are outer Kleinians obeying

$$k \star f = \pi(f) \star k, \quad \bar{k} \star f = \bar{\pi}(f) \star \bar{k}, \quad k \star k = 1 = \bar{k} \star \bar{k}, \quad (2.4)$$

with automorphisms π and $\bar{\pi}$ defined by $\pi d = d\pi$, $\bar{\pi} d = d\bar{\pi}$ and

$$\begin{aligned} \pi(f(z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}})) &= f(-z^\alpha, \bar{z}^{\dot{\alpha}}; -y^\alpha, \bar{y}^{\dot{\alpha}}), \\ \bar{\pi}(f(z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}})) &= f(z^\alpha, -\bar{z}^{\dot{\alpha}}; y^\alpha, -\bar{y}^{\dot{\alpha}}). \end{aligned} \quad (2.5)$$

The bosonic projection condition amounts to

$$\pi \bar{\pi}(f) = f, \quad f = P_+ \star f, \quad P_\pm := \frac{1}{2}(1 \pm k \star \bar{k}), \quad (2.6)$$

which implies

$$f = \left[f^{(+)}(X, dX; Z, dZ; Y) + f^{(-)}(X, dX; Z, dZ; Y) \star \frac{(k + \bar{k})}{2} \right] \star P_+. \quad (2.7)$$

The bosonic projection removes all component fields associated with the unfolding of spinorial degrees of freedom in spacetime. Irreducible minimal bosonic models can be obtained by imposing reality conditions and discrete symmetries that remove all odd spins; the hermitian conjugation \dagger and the relevant anti-automorphism τ are defined by $d((\cdot)^\dagger) = (d(\cdot))^\dagger$, $d\tau = \tau d$ and

$$(f(z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}}; k, \bar{k}))^\dagger = \bar{f}(\bar{z}^{\dot{\alpha}}, z^\alpha; \bar{y}^{\dot{\alpha}}, y^\alpha; \bar{k}, k), \quad (2.8)$$

$$\tau(f(z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}}; k, \bar{k})) = f(-iz^\alpha, -i\bar{z}^{\dot{\alpha}}; iy^\alpha, i\bar{y}^{\dot{\alpha}}; k, \bar{k}), \quad (2.9)$$

$$(f_{[p]} \star f'_{[p']})^\dagger = (-1)^{pp'} (f'_{[p']})^\dagger \star (f_{[p]})^\dagger, \quad \tau(f_{[p]} \star f'_{[p']})^\dagger = (-1)^{pp'} \tau(f'_{[p']}) \star \tau(f_{[p]}). \quad (2.10)$$

We shall discuss the minimal models below.

The duality extension of the Vasiliev system is based on the following generalized curvature constraints

$$F + \mathcal{F} = 0, \quad DB = 0, \quad (2.11)$$

with Yang–Mills-like curvature and covariant derivative defined by

$$F = dA + A \star A, \quad DB = dB + A \star B - B \star A, \quad (2.12)$$

and interaction freedom ($I, \bar{I} = 1, 2$)

$$\mathcal{F} = \mathcal{F}_I(B) \star J_{[2]}^I + \mathcal{F}_{\bar{I}}(B) \star J_{[2]}^{\bar{I}} + \mathcal{F}_{I\bar{I}}(B) \star J_{[4]}^{I\bar{I}} \quad (2.13)$$

featuring the central elements

$$(J_{[2]}^I)_{I=1,2} = -\frac{i}{4}(1, k\kappa) \star P_+ \star d^2 z, \quad (J_{[2]}^{\bar{I}})_{\bar{I}=\bar{1},\bar{2}} = -\frac{i}{4}(1, \bar{k}\bar{\kappa}) \star P_+ \star d^2 \bar{z}, \quad (2.14)$$

$$J_{[4]}^{I\bar{I}} = -\frac{i}{4}J_{[2]}^I J_{[2]}^{\bar{I}}, \quad (2.15)$$

and \star -functions \mathcal{F}_I , $\mathcal{F}_{\bar{I}}$ and $\mathcal{F}_{I\bar{I}}$ of B such that $\mathcal{F}_I(\lambda)$, $\mathcal{F}_{\bar{I}}(\lambda)$ and $\mathcal{F}_{I\bar{I}}(\lambda)$ ($I, \bar{I} = 1, 2$), viewed as functions of a single complex variable $\lambda \in \mathbb{C}$, are complex analytic in a finite neighborhood of $\lambda = 0$.

The unfolded equations (2.11) are Cartan integrable because the Yang–Mills-like Bianchi identities $DF \equiv 0$ and $DDB \equiv [F, B]_\star$ are compatible with the generalized curvature constraints. In other words, defining the generalized curvatures

$$\mathcal{R}^A = F + \mathcal{F}, \quad \mathcal{R}^B = DB, \quad (2.16)$$

one has the generalized Bianchi identities

$$D\mathcal{R}^A - (\mathcal{R}^B \partial_B) \star \mathcal{F} \equiv 0, \quad D\mathcal{R}^B - [\mathcal{R}^A, B]_\star \equiv 0. \quad (2.17)$$

The potentials $\{A_{[1]}, B_{[2]}, A_{[3]}, B_{[4]}, \dots\}$ in positive form degree share one and the same Weyl zero-form $B_{[0]}$, that hence contain all the local perturbative degrees of freedom of the extended system. One may refer to $\{B_{[0]}, A_{[1]}, B_{[2]}, A_{[3]}, B_{[4]}, \dots\}$ as a duality extension of the original Vasiliev system consisting of $\{B_{[0]}, A_{[1]}\}$ in the sense that the presence of the central elements in degree four implies that $\{B_{[2]}, A_{[3]}, B_{[4]}, \dots\}$ cannot in general be set equal to zero on shell. Moreover, the extension is massless in the sense that for each $p \in \{1, 2, 3, \dots\}$ the system of forms with degrees $p' \leq p$ constitutes a closed subsystem, *i.e.* their curvatures do not depend on the forms with degrees $p' > p$. In particular, this means that any (locally defined) exact solution to the duality extended system contains a (locally defined) exact solution to the original Vasiliev system. The converse statement requires a more careful analysis that we defer here.

2.2 A duality extended spectral flow

The duality extended system possesses a spectral flow [34] describing the evolution of the system on shell under changes in a vacuum expectation value ν and a coupling g defined by the field redefinition

$$B = \nu \mathbf{1} + gB'. \quad (2.18)$$

We stress that the parameters (g, ν) are part of the moduli space of the unfolded equations of motion, that is, both A and B depend on (g, ν) on shell and in such a way that d commutes with $(\partial_g, \partial_\nu)$. Letting $f = f(A, dA, B, dB)$ and defining the flow operator

$$L_1 f = \partial_g f - \mu_1 B' \star \partial_\nu f - \partial_\nu f \star \mu_2 B', \quad \mu_1, \mu_2 \in \mathbb{C}, \quad \mu_1 + \mu_2 = 1, \quad (2.19)$$

one has

$$L_1 F \equiv DL_1 A + \mu_1 DB' \star \partial_\nu A - \mu_2 \partial_\nu A \star DB', \quad (2.20)$$

$$L_1 DB \equiv DL_1 B + [L_1 A, B]_\star + \mu_1 DB \star \partial_\nu B' + \mu_2 \partial_\nu B' \star DB, \quad (2.21)$$

$$L_1 \mathcal{F} \equiv (L_1 B \partial_B) \star \mathcal{F}. \quad (2.22)$$

It follows that the duality extended equations of motion are compatible with the flow equations

$$L_1 A \approx 0, \quad L_1 B \approx 0, \quad (2.23)$$

where the last flow equation is equivalent to that $L_1 B' \approx 0$.

The flow equations generalize as follows: one first redefines

$$B = \nu + \mathcal{N}(B'), \quad \mathcal{N} = \nu_1 g B' + \nu_2 g^2 B'^{\star 2} + \nu_3 g^3 B'^{\star 3} + \dots, \quad (2.24)$$

where ν_k ($k \geq 1$) are constants and g the coupling. The flow operator defined by

$$L f = \partial_g f - \mathcal{M}_1(B') \star \partial_\nu f - \partial_\nu f \star \mathcal{M}_2(B'), \quad (2.25)$$

where the two \star -functions defined by ($i = 1, 2$)

$$\mathcal{M}_i = \mu_{i,1} g B' + \mu_{i,2} g^2 B'^{\star 2} + \dots, \quad \mu_{1,k} + \mu_{2,k} = k \nu_k \quad (k \geq 1); \quad (2.26)$$

obey

$$L \mathcal{F} \equiv (LB \partial_B) \star \mathcal{F}, \quad (2.27)$$

$$LB = \nu_1 L B' + \nu_2 g^2 (L B' \star B' + B' \star L B') + \dots, \quad (2.28)$$

$$L F = D L A + D \mathcal{M}_1 \star \partial_\nu A - \partial_\nu A \star D \mathcal{M}_2, \quad (2.29)$$

$$L D B' = D L B' + [L A, B']_\star + D \mathcal{M}_1 \star \partial_\nu B' + \partial_\nu B' \star D \mathcal{M}_2, \quad (2.30)$$

and it follows that one can set the constraints

$$LA = 0, \quad LB' = 0, \quad (2.31)$$

where the latter constraint thus implies that $LB = 0$. One can redefine $\mathcal{N} = gB'$ so that $\nu_1 = 1$ and $\nu_k = 0$ for $k > 1$, leaving the freedom in \mathcal{M}_i that generalizes the two-parameter freedom in having μ_1 and μ_2 .

2.3 Consistent truncations

There are two possible reality conditions leading to models with negative cosmological constant $\Lambda < 0$, that we parameterize using $\epsilon_{\mathbb{R}} = \pm 1$ as follows:

$$(A_{[p]})^\dagger = -(\epsilon_{\mathbb{R}})^{\frac{p-1}{2}} A_{[p]}, \quad (B_{[p]})^\dagger = (\epsilon_{\mathbb{R}})^{\frac{p}{2}} B_{[p]}, \quad (2.32)$$

$$(\mathcal{F}_I(\lambda))^\dagger = \mathcal{F}_{\bar{I}}(\lambda^\dagger), \quad \mathcal{F}_{I\bar{J}}(\lambda)^\dagger = \epsilon_{\mathbb{R}} \mathcal{F}_{J\bar{I}}(\lambda^\dagger). \quad (2.33)$$

Moreover, using the map

$$\pi_k : (k, \bar{k}) \mapsto (-k, -\bar{k}), \quad (2.34)$$

there are two possible projections to models without topological (adjoint) zero-forms, that we parameterize using $\epsilon_k = \pm 1$ as follows:

$$\pi_k(A_{[p]}) = (\epsilon_k)^{\frac{p-1}{2}} A_{[p]}, \quad \pi_k(B_{[p]}) = -(\epsilon_k)^{\frac{p}{2}} B_{[p]}, \quad (2.35)$$

$$\mathcal{F}_I(-\lambda) = (-1)^{I+1} \mathcal{F}_I(\lambda), \quad \mathcal{F}_{I\bar{I}}(-\lambda) = (-1)^{I+\bar{I}} \epsilon_k \mathcal{F}_{I\bar{I}}(\lambda). \quad (2.36)$$

Using the parity transformation P defined by $Pd = dP$ and

$$P(f(X^M; z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}}; k, \bar{k})) = (Pf)(X^M; -\bar{z}^{\dot{\alpha}}, -z^\alpha; \bar{y}^{\dot{\alpha}}, y^\alpha; \bar{k}, k), \quad (2.37)$$

which is an automorphism of the \star -product algebra and where Pf is expanded in terms of parity reversed component fields, there are four ways of fixing parities, that we parameterize using $\epsilon, \tilde{\epsilon} = \pm 1$ as follows:

$$P(A_{[p]}) = (\epsilon \tilde{\epsilon})^{\frac{p-1}{2}} A_{[p]}, \quad P(B_{[p]}) = (\epsilon)^{\frac{p+2}{2}} (\tilde{\epsilon})^{\frac{p}{2}} B_{[p]}, \quad (2.38)$$

$$\mathcal{F}_{\bar{I}}(\lambda) = \mathcal{F}_I(\epsilon \lambda), \quad \mathcal{F}_{I\bar{J}}(\lambda) = \epsilon \tilde{\epsilon} \mathcal{F}_{J\bar{I}}(\epsilon \lambda). \quad (2.39)$$

Finally, the τ -projection to the minimal models with only even propagating spins reads

$$\tau(A_{[p]}) = (-1)^{\frac{p+1}{2}} A_{[p]}, \quad \tau(B_{[p]}) = (-1)^{\frac{p}{2}} B_{[p]}, \quad (2.40)$$

which is the unique choice since $\tau(J_{[p]}) = (-1)^{\frac{p}{2}} J_{[p]}$ (and there is no condition on \mathcal{F}).

In the $(B_{[0]}, A_{[1]})$ -sector, which forms a closed subsystem, the assignment of k -parity combined with the freedom in redefining A_α can be used to replace [2]

$$(\mathcal{F}_1, \mathcal{F}_2; \mathcal{F}_{\bar{1}}, \mathcal{F}_{\bar{2}}) \rightarrow (0, (1 - \mathcal{F}_1)^{\star(-1)} \star \mathcal{F}_2; 0, (1 - \mathcal{F}_{\bar{1}})^{\star(-1)} \star \mathcal{F}_{\bar{2}}). \quad (2.41)$$

Imposing also reality and parity conditions, of which the latter is a multiple choice parametrized by $\epsilon = \pm 1$, the remaining interaction function $(1 - \mathcal{F}_1)^{\star(-1)} \star \mathcal{F}_2$ becomes real and odd, hence defining the new master field

$$\Phi \star P_+ := (1 - \mathcal{F}_1)^{\star(-1)} \star \mathcal{F}_2 \star k \star P_+, \quad (2.42)$$

obeying the twisted reality condition $(\Phi)^\dagger = \pi(\Phi)$ and the parity condition $P(\Phi) = \epsilon\Phi$ leading to a physical scalar that is even under parity for $\epsilon = 1$ and odd under parity for $\epsilon = -1$. Finally, one may project out the odd spins by imposing $\tau(\Phi) = \pi(\Phi)$ yielding the minimal bosonic models.

Assuming linear interaction functions

$$\mathcal{F}_I = b_I B, \quad \mathcal{F}_{\bar{I}} = b_{\bar{I}} B, \quad \mathcal{F}_{I\bar{I}} = c_{I\bar{I}} B, \quad (2.43)$$

and defining a total central element

$$J = J_{[2]} + J_{[4]} \quad (2.44)$$

via

$$B \star J_{[2]} = \mathcal{F}_I \star J_{[2]}^I + \mathcal{F}_{\bar{I}} \star J_{[2]}^{\bar{I}}, \quad B \star J_{[4]} = \mathcal{F}_{I\bar{I}} \star J_{[4]}^{I\bar{I}}, \quad (2.45)$$

$$J_{[2]} = -\frac{i}{4} [dz^2(b_1 + b_2 k \kappa) + d\bar{z}^2(b_{\bar{1}} + b_{\bar{2}} \bar{k} \bar{\kappa})] \star P_+, \quad (2.46)$$

$$J_{[4]} = -\frac{i}{4} dz^2 d\bar{z}^2 (c_{1\bar{1}} + c_{2\bar{1}} k \kappa + c_{1\bar{2}} \bar{k} \bar{\kappa} + c_{2\bar{2}} \kappa \bar{\kappa}) \star P_+, \quad (2.47)$$

the reality, k -parity and P -parity conditions imply

$$(J_{[p]})^\dagger = -(\epsilon_{\mathbb{R}})^{\frac{p-2}{2}} J_{[p]}, \quad \pi_k(J_{[p]}) = -(\epsilon_k)^{\frac{p-2}{2}} J_{[p]}, \quad P(J_{[p]}) = (\epsilon)^{\frac{p}{2}} (\tilde{\epsilon})^{\frac{p-2}{2}} J_{[p]}, \quad (2.48)$$

which constrain the parameters $(b_I, b_{\bar{I}}, c_{I\bar{I}})$. These conditions admit nontrivial solutions for $J_{[p]}$ for all combinations of signs except for $\epsilon_k = \tilde{\epsilon} = -1$ since $\epsilon_k = -1$ implies that $\tilde{\epsilon} = +1$.

3 Generalized Hamiltonian action principle

3.1 Graded cyclic chiral trace

Vasiliev's equations are formulated in terms of master fields which one may think of as functions on a total space called *correspondance space* \mathfrak{C} , that is locally a product space $M_\xi \times \mathfrak{Z} \times \mathfrak{Y}$ where \mathfrak{Z} and \mathfrak{Y} are two copies of a non-commutative twistor space and M_ξ denotes a coordinate chart of a commuting base manifold M , see Appendix D for more details. In order to build an action principle, we need to integrate over the correspondance space. The integration over \mathfrak{C} of a globally defined $(\hat{p} + 1)$ -form \mathcal{L} is defined by

$$\int_{\mathfrak{C}} \mathcal{L} = \sum_{\xi} \int_{M_\xi} \text{Tr} [f_{\mathcal{L}}] , \quad (3.1)$$

where $f_{\mathcal{L}}$ denotes a symbol of \mathcal{L} and the chiral trace operation is defined by

$$\text{Tr} [f] = \sum_m \int_{\mathfrak{Z} \times \mathfrak{Y}} \frac{d^2 y d^2 \bar{y}}{(2\pi)^2} \frac{f_{[m;2,2]}|_{k=0=\bar{k}}}{(2\pi)^2} , \quad (3.2)$$

using the decomposition $f_{[p]} = \sum_{\substack{m+q+\bar{q}=p \\ q, \bar{q} \leq 2}} f_{[m;q,\bar{q}]}$ with

$$f_{[m;q,\bar{q}]}(\lambda dX^M; \mu dz^\alpha, \bar{\mu} d\bar{z}^{\dot{\alpha}}) = \lambda^m \mu^q \bar{\mu}^{\bar{q}} f_{[m;q,\bar{q}]}(dX^M; dz^\alpha, d\bar{z}^{\dot{\alpha}}) , \quad (3.3)$$

and with integration domain consisting of real contours for $\{y^\alpha, z^\alpha\}$ and $\{\bar{y}^{\dot{\alpha}}, \bar{z}^{\dot{\alpha}}\}$, respectively, that is, one performs separate integrations over the holomorphic and anti-holomorphic variables treated as independent real variables (for related discussions, see *e.g.* Appendix G of [35]). The choice of the chiral integration domain (instead of the complex integration domain) implies that

$$\text{Tr} [\pi(f)] = \text{Tr} [\bar{\pi}(f)] = \text{Tr} [f] , \quad (3.4)$$

which in its turn implies graded cyclicity,

$$\text{Tr} [f_{[p]} \star f'_{[p']}] = (-1)^{pp'} \text{Tr} [f'_{[p']} \star f_{[p]}] , \quad (3.5)$$

as can be seen by expanding $f_{[p]} = (f_{[p]}^{(+)} + f_{[p]}^{(-)} \star k) \star P_+ \text{idem } f'_{[p']}$ which yields

$$\text{Tr} [f_{[p]} \star f'_{[p']}] = \frac{1}{2} \text{Tr} [f_{[p]}^{(+)} \star f'_{[p']}^{(+)} + f_{[p]}^{(-)} \star \pi(f'_{[p']}^{(-)})] , \quad (3.6)$$

where the second term is graded cyclic by virtue of the chiral integration. Furthermore, the chiral trace operation commutes to hermitian conjugation and is invariant under P and π_k ,

$$(\text{Tr} [f])^\dagger = \text{Tr} [(f)^\dagger] , \quad \text{Tr} [P(f)] = \text{Tr} [f] , \quad \text{Tr} [\pi_k(f)] = \text{Tr} [f] . \quad (3.7)$$

Finally, one may seek to impose boundary conditions in $\mathfrak{Z} \times \mathfrak{Y}$ such that the integration contours can be rotated from real to imaginary axes in the sense that

$$\mathrm{Tr} [\tau(f)] = \mathrm{Tr} [f] . \quad (3.8)$$

We shall finally assume that the integration over \mathfrak{C} is non-degenerate such that if $\mathrm{Tr} [f \star g] = 0$ for all f then $g = 0$. It is an interesting open problem to understand whether the π , P and τ symmetries could be violated on classical observables evaluated on exact solutions that one may seek to interpret as describing topology changes of the twistor space which we leave for future studies [36]. In what follows, we shall always assume that the discrete symmetries hold off shell.

3.2 Odd-dimensional bulk ($\hat{p} \in 2\mathbb{N}$)

3.2.1 Action principle

In the case of an odd-dimensional base manifold of dimension $\hat{p}+1 = 2n+5$ with $n \in \{0, 1, 2, \dots\}$ such that $\dim(M) = 2n+1$, the duality-extended system equations of motion follows from the variational principle based on the generalized Hamiltonian bulk action

$$S_{\mathrm{bulk}}^{\mathrm{cl}}[\{A, B, U, V\}_\xi] = \sum_\xi \int_{M_\xi} \mathrm{Tr} \left[U \star DB + V \star \left(F + \mathcal{G}(B, U; J^I, J^{\bar{I}}, J^{I\bar{I}}) \right) \right] , \quad (3.9)$$

with interaction freedom \mathcal{G} and locally defined master fields decomposing under total form degree into

$$A = A_{[1]} + A_{[3]} + \dots + A_{[2n-1]} , \quad B = B_{[0]} + B_{[2]} + \dots + B_{[2n-2]} , \quad (3.10)$$

$$U = U_{[2]} + U_{[4]} + \dots + U_{[2n]} , \quad V = V_{[1]} + V_{[3]} + \dots + V_{[2n-1]} . \quad (3.11)$$

The function \mathcal{G} must be constrained in order for the action to be gauge invariant and in order to avoid systems that are trivial. In what follows, we shall consider the special case

$$\mathcal{G} = \mathcal{F}(B; J^I, J^{\bar{I}}, J^{I\bar{I}}) + \widetilde{\mathcal{F}}(U; J^I, J^{\bar{I}}, J^{I\bar{I}}) , \quad (3.12)$$

$$\mathcal{F} = \mathcal{F}_I(B) \star J_{[2]}^I + \mathcal{F}_{\bar{I}}(B) \star J_{[2]}^{\bar{I}} + \mathcal{F}_{I\bar{I}}(B) \star J_{[4]}^{I\bar{I}} , \quad (3.13)$$

$$\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_0(U) + \widetilde{\mathcal{F}}_I(U) \star J_{[2]}^I + \widetilde{\mathcal{F}}_{\bar{I}}(U) \star J_{[2]}^{\bar{I}} + \widetilde{\mathcal{F}}_{I\bar{I}}(U) \star J_{[4]}^{I\bar{I}} , \quad (3.14)$$

where $\partial_U \widetilde{\mathcal{F}}_0|_{U=0}$ must vanish (since else the bulk action can be brought to the trivial form with $\mathcal{G} = U$ modulo a total derivative).

Denoting $Z^i = (A, B, U, V)$, the general variation of the action defines generalized curvatures \mathcal{R}^i as follows:

$$\delta S = \sum_{\xi} \int_{M_{\xi}} \text{Tr} [\mathcal{R}^i \star \delta Z^j \mathcal{O}_{ij}] + \sum_{\xi} \int_{\partial M_{\xi}} \text{Tr} [U \star \delta B - V \star \delta A], \quad (3.15)$$

where one thus has

$$\mathcal{R}^A = F + \mathcal{F} + \widetilde{\mathcal{F}}, \quad \mathcal{R}^B = DB + (V \partial_U) \star \widetilde{\mathcal{F}}, \quad (3.16)$$

$$\mathcal{R}^U = DU - (V \partial_B) \star \mathcal{F}, \quad \mathcal{R}^V = DV + [B, U]_{\star}, \quad (3.17)$$

with \mathcal{O}_{ij} being a constant non-degenerate matrix (defining a symplectic form of degree $\hat{p} + 2$ on the \mathbb{N} -graded target space of the bulk theory). Treating Z^i and dZ^i as independent variables, one has the differential identities

$$D\mathcal{R}^A - (\mathcal{R}^B \partial_B) \star \mathcal{F} - (\mathcal{R}^U \partial_U) \star \widetilde{\mathcal{F}} \equiv \mathcal{A}^A, \quad (3.18)$$

$$D\mathcal{R}^B - [\mathcal{R}^A, B]_{\star} - (\mathcal{R}^V \partial_U) \star \widetilde{\mathcal{F}} - (\mathcal{R}^U \partial_U) \star (V \partial_U) \star \widetilde{\mathcal{F}} \equiv \mathcal{A}^B, \quad (3.19)$$

$$D\mathcal{R}^U - [\mathcal{R}^A, U]_{\star} + (\mathcal{R}^V \partial_B) \star \mathcal{F} + (\mathcal{R}^B \partial_B) \star (V \partial_B) \star \widetilde{\mathcal{F}} \equiv \mathcal{A}^U, \quad (3.20)$$

$$D\mathcal{R}^V - [\mathcal{R}^A, V]_{\star} - [\mathcal{R}^B, U]_{\star} + [\mathcal{R}^U, B]_{\star} \equiv \mathcal{A}^V, \quad (3.21)$$

with dZ^i -independent quantities $\mathcal{A}^i \equiv \mathcal{A}^i(Z^j)$ given by

$$\mathcal{A}^A \equiv -((V \partial_U) \star \widetilde{\mathcal{F}}) \partial_B \star \mathcal{F} + ((V \partial_B) \star \mathcal{F}) \partial_U \star \widetilde{\mathcal{F}}, \quad (3.22)$$

$$\mathcal{A}^B \equiv ((V \partial_B) \star \mathcal{F}) \partial_U \star (V \partial_U) \star \widetilde{\mathcal{F}}, \quad (3.23)$$

$$\mathcal{A}^U \equiv ((V \partial_U) \star \widetilde{\mathcal{F}}) \partial_B \star (V \partial_B) \star \mathcal{F}, \quad (3.24)$$

$$\mathcal{A}^V \equiv 0, \quad (3.25)$$

where the last identity follows from

$$[U, (V \partial_U) \star \widetilde{\mathcal{F}}]_{\star} \equiv -[V, \widetilde{\mathcal{F}}]_{\star}, \quad [B, (V \partial_B) \star \mathcal{F}]_{\star} \equiv -[V, \mathcal{F}]_{\star}. \quad (3.26)$$

The quantities \mathcal{A}^i thus represent obstructions to generalized Bianchi identities off shell and hence to Cartan integrability of the unfolded equations of motion $\mathcal{R}^i \approx 0$, where in this Section we use weak

equalities for equations that hold on shell. These obstructions vanish identically (without further algebraic constraints on Z^i) in at least the following two cases:

$$\text{bilinear } Q\text{-structure} \quad : \quad \mathcal{F} = B \star J, \quad J = J_{[2]} + J_{[4]}, \quad (3.27)$$

$$\text{bilinear } P\text{-structure} \quad : \quad \widetilde{\mathcal{F}} = U \star J', \quad J' = J'_{[2]} + J'_{[4]}, \quad (3.28)$$

where the central elements are expanded as in Eqs. (2.44)–(2.47).

At this stage it is useful to recall (see Appendix C) that if $\mathcal{R}^i = dZ^i + \mathcal{Q}^i(Z^j)$ defines a set of generalized curvatures, then one has the following three equivalent statements: i) \mathcal{R}^i obey a set of generalized Bianchi identities $d\mathcal{R}^i - (\mathcal{R}^j \partial_j) \star \mathcal{Q}^i \equiv 0$; ii) \mathcal{R}^i transform into each other under Cartan gauge transformations $\delta_\varepsilon Z^i = d\varepsilon^i - (\varepsilon^j \partial_j) \star \mathcal{Q}^i$; and iii) the quantity $\overrightarrow{\mathcal{Q}} := \mathcal{Q}^i \partial_i$ is a Q -structure, *i.e.* a nilpotent \star -vector field of degree one in target space, *viz.* $\overrightarrow{\mathcal{Q}} \star \mathcal{Q}^i \equiv 0$. Furthermore, in the case of differential algebras on commutative base manifolds, one can show that if \mathcal{R}^i are defined via a variational principle as in (3.15) (with constant \mathcal{O}_{ij}), then the action S remains invariant under $\delta_\varepsilon Z^i$.

In the two Cartan integrable cases at hand, one thus has the on-shell Cartan gauge transformations

$$\delta_{\varepsilon, \eta} A = D\varepsilon^A - (\varepsilon^B \partial_B) \star \mathcal{F} - (\eta^U \partial_U) \star \widetilde{\mathcal{F}}, \quad (3.29)$$

$$\delta_{\varepsilon, \eta} B = D\varepsilon^B - [\varepsilon^A, B]_\star - (\eta^V \partial_U) \star \widetilde{\mathcal{F}} - (\eta^U \partial_U) \star (V \partial_U) \star \widetilde{\mathcal{F}}, \quad (3.30)$$

$$\delta_{\varepsilon, \eta} U = D\eta^U - [\varepsilon^A, U]_\star + (\eta^V \partial_B) \star \mathcal{F} + (\varepsilon^B \partial_B) \star (V \partial_B) \star \mathcal{F}, \quad (3.31)$$

$$\delta_{\varepsilon, \eta} V = D\eta^V - [\varepsilon^A, V]_\star - [\varepsilon^B, U]_\star + [\eta^U, B]_\star. \quad (3.32)$$

These transformations remain symmetries off shell as can be seen using the following set of identities:

$$\text{bilinear } P\text{-structure} \quad : \quad \text{Tr} [J' \star V \star (V \partial_B) \star (\varepsilon^B \partial_B) \star \mathcal{F}] \equiv 0, \quad (3.33)$$

$$\begin{aligned} \text{Tr} [V \star (DB \partial_B) \star (\varepsilon^B \partial_B) \star \mathcal{F} + DB \star (V \partial_B) \star (\varepsilon^B \partial_B) \star \mathcal{F}] &\equiv 0, \\ \text{Tr} [\eta^V \star (DB \partial_B) \star \mathcal{F} - DB \star (\eta^V \partial_B) \star \mathcal{F}] &\equiv 0, \end{aligned} \quad (3.34)$$

$$\text{bilinear } Q\text{-structure} \quad : \quad \text{Tr} [J \star V \star (V \partial_U) \star (\eta^U \partial_U) \star \widetilde{\mathcal{F}}] \equiv 0, \quad (3.35)$$

$$\begin{aligned} \text{Tr} [V \star (DU \partial_U) \star (\eta^U \partial_U) \star \widetilde{\mathcal{F}} + DU \star (V \partial_U) \star (\eta^U \partial_U) \star \widetilde{\mathcal{F}}] &\equiv 0, \\ \text{Tr} [\eta^V \star (DU \partial_U) \star \widetilde{\mathcal{F}} - DU \star (\eta^V \partial_U) \star \widetilde{\mathcal{F}}] &\equiv 0. \end{aligned} \quad (3.36)$$

More precisely, the $(\epsilon^A; \epsilon^B)$ -symmetries leave the Lagrangian invariant while the (η^U, η^V) -symmetries transform the Lagrangian into a nontrivial total derivative, *viz.*

$$\delta_{\epsilon, \eta} \mathcal{L} \equiv d \left(\text{Tr} \left[\eta^U \star \mathcal{K}_U + \eta^V \star \mathcal{K}_V \right] \right), \quad (3.37)$$

for $(\mathcal{K}_U, \mathcal{K}_V)$ that are not identically zero. It follows that the Cartan gauge algebra \mathfrak{g} is of the form

$$\mathfrak{g} \cong \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with $\mathfrak{g}_1 \cong \text{span}\{\epsilon^A, \epsilon^B\}$ and $\mathfrak{g}_2 \cong \text{span}\{\eta^U, \eta^V\}$, as one can verify explicitly using the formulae (C.15) given in Appendix C.

3.2.2 Global formulation, boundary conditions and embedding of Vasiliev's original system

Exponentiation of the infinitesimal Cartan gauge transformations leads to locally defined gauge orbits consisting of elements (see Appendix A)

$$Z_{\lambda, d\lambda; Z_0}^i = \mathcal{G}_{\lambda, d\lambda; Z} \star Z^i|_{Z^i=Z_0^i}, \quad (3.38)$$

$$\mathcal{G}_{\lambda, d\lambda; Z} := \exp_\star \overrightarrow{\mathcal{F}}_{\lambda, d\lambda; Z}, \quad \overrightarrow{\mathcal{F}}_{\lambda, d\lambda; Z} := (d\lambda^i - (\lambda^j \partial_j) \star \mathcal{Q}^i) \frac{\partial}{\partial Z^i}, \quad (3.39)$$

where λ^i and Z_0^i , respectively, are gauge functions and representatives of the orbits defined in coordinate charts of the base manifold. On shell, one has

$$dZ_0^i + \mathcal{Q}^i(Z_0^j) \approx 0 \quad \Rightarrow \quad dZ_{\lambda, d\lambda; Z_0}^i + \mathcal{Q}^i(Z_{\lambda, d\lambda; Z_0}^j) \approx 0, \quad (3.40)$$

as can be seen by first writing $d \approx \overrightarrow{\mathcal{F}}_{d\lambda} - \overrightarrow{\mathcal{Q}}$ where $\overrightarrow{\mathcal{F}}_{d\lambda} := d\lambda^i \partial / \partial \lambda^i$ and $\overrightarrow{\mathcal{Q}} := \mathcal{Q}^i \partial / \partial Z^i$, and then using $[\overrightarrow{\mathcal{F}}_{d\lambda} - \overrightarrow{\mathcal{Q}}, \overrightarrow{\mathcal{F}}_{\lambda, d\lambda; Z}]_\star \equiv 0$ and $[\exp_\star \overrightarrow{\mathcal{X}}]_\star (\mathcal{F} \star \mathcal{F}') \equiv ([\exp_\star \overrightarrow{\mathcal{X}}]_\star \mathcal{F}) \star ([\exp_\star \overrightarrow{\mathcal{X}}]_\star \mathcal{F}')$ for \star -vector fields $\overrightarrow{\mathcal{X}}$ and \star -functions \mathcal{F} and \mathcal{F}' (see Appendix C for details).

In particular, it follows that the space of (locally defined) classical solutions to the duality extended $(A, B; U, V)$ -system contains a subspace of (locally defined) classical solutions to the duality extended (A, B) -system, obtained simply by setting $U = V = 0$. The (A, B) -system contains in its turn a subset of the (locally defined) solutions to the original Vasiliev system in form degrees 0 and 1. The converse issue, whether any given (locally defined) exact solution to the original Vasiliev system can be uplifted to the (A, B) -system, requires, however, a more careful analysis of the gauge orbits in degrees greater than 1 (due to the non-polynomial dependencies on the integration constants for the Weyl zero-form and the zero-form gauge functions).

Turning to the global formulation, it follows from Eq. (3.37) that the gauge parameters $(\epsilon_\xi^A, \epsilon_\xi^B) \in \mathfrak{g}_1$ can be locally defined on M , that is, defined independently on the coordinate charts M_ξ (provided that the action is not perturbed by impurities that break some of the (ϵ^A, ϵ^B) -symmetries, as for example in the soldered phase where perturbations break the local translations in $\epsilon^{A[1]}$). From Eq. (3.37) it also follows that $(\eta^U, \eta^V) \in \mathfrak{g}_2$ need to be defined globally on M , that is, $(\eta^U, \eta^V)|_\xi$ and $(\eta^U, \eta^V)|_{\xi'}$ must be related by transition functions across the chart boundary between M_ξ and $M_{\xi'}$ (in practice this means that one may take (η_ξ^U, η_ξ^V) to have compact support in M_ξ).

The unbroken phase of the theory thus consists of local representatives $Z_\xi^i = (A, B; U, V)|_\xi$ defined up to gauge transformations with parameters $(\epsilon_\xi^A; \epsilon_\xi^B)$ that are unrestricted on ∂M_ξ and parameters (η_ξ^U, η_ξ^V) with the aforementioned restrictions on ∂M_ξ , with transitions of the form

$$Z_\xi^i = \mathcal{G}_\xi^{\xi'} \star Z_{\xi'}^i \quad \text{defined on } M_\xi \cap M_{\xi'} . \quad (3.41)$$

where $\mathcal{G}_\xi^{\xi'} = \exp \mathcal{T}_{\lambda; Z}|_\xi^{\xi'}$ with gauge functions $\lambda_\xi^{\xi'} \in \mathfrak{g}_1$ defined on $M_\xi \cap M_{\xi'}$.

More generally, softly broken (homotopy) phases of the theory arise by taking the transition functions to be generated by various unbroken subalgebras $\mathfrak{t} \subseteq \mathfrak{g}_1$. Their moduli spaces $\mathcal{M}_\mathfrak{t}$ can be coordinatized by classical observables $\mathcal{O}_\mathfrak{t}$ that are manifestly \mathfrak{t} -invariant off shell and diffeomorphism invariant on shell (one may thus think the unbroken phase $\mathcal{M}_\mathfrak{g}$ as the smallest homotopy phase for a given base manifold; it can be embedded into various broken phases). Of particular interest is the soldered phase in which the action is perturbed as to softly break the gauge symmetries associated to the π -odd projection of $A_{[1]}$. The unbroken gauge algebra in this case thus consists of the π -even projection $\frac{1}{2}(1 + \pi)\epsilon^{A[1]}$ together with the remaining ϵ -parameters of positive form degree.

Hence, to achieve a globally well-defined variational principle, one considers globally defined field configurations off shell consisting of locally defined representatives Z_ξ^i related on chart boundaries via transitions (3.41) for a given structure algebra $\mathfrak{t} \subseteq \mathfrak{g}_1$. The manifest \mathfrak{g}_1 -invariance implies that in the general variation (3.15), the contributions from two adjacent boundaries ∂M_ξ and $\partial M_{\xi'}$ cancel; on such

a boundary one has the transition functions ($\epsilon \equiv \epsilon_{\xi}^{\xi'}$)

$$\delta_{\epsilon}(\delta A) = -[\epsilon^A, \delta A]_{\star} - (\delta B \partial_B) \star (\epsilon^B \partial_B) \star \mathcal{F}, \quad (3.42)$$

$$\delta_{\epsilon}(\delta B) = -[\epsilon^A, \delta B]_{\star} + \{\epsilon^B, \delta A\}_{\star}, \quad (3.43)$$

$$\delta_{\epsilon}U = -[\epsilon^A, U]_{\star} + (\epsilon^B \partial_B) \star (V \partial_B) \star \mathcal{F}, \quad (3.44)$$

$$\delta_{\epsilon}V = -[\epsilon^A, V]_{\star} - [\epsilon^B, U]_{\star}, \quad (3.45)$$

which implies that ($\epsilon \equiv \epsilon_{\xi}^{\xi'}$)

$$\delta_{\epsilon} \left(\int_{\partial M_{\xi}} \text{Tr} [U \star \delta B - V \star \delta A] \right) \quad (3.46)$$

$$= \int_{\partial M_{\xi}} \text{Tr} [V \star (\delta B \partial_B) \star (\epsilon^B \partial_B) \star \mathcal{F} - \delta B \star (V \partial_B) \star (\epsilon^B \partial_B) \star \mathcal{F}] \equiv 0. \quad (3.47)$$

One is thus left with contributions from true boundaries $\partial M_{\xi} \subset \partial M$ (including boundaries of homotopy cylinders surrounding impurities of co-dimension greater than one). It follows that the natural boundary conditions compatible with the locally defined gauge symmetries are the Dirichlet conditions

$$(U, V)|_{\partial M} = 0. \quad (3.48)$$

In summary, a classical solution can thus be specified by fixing i) the transition functions; ii) an initial datum for the zero-form $B_{[0]}$, say

$$B_{[0]}|_p = C(Y; k, \bar{k}), \quad (3.49)$$

at some given point $p \in \mathfrak{B}$ in the base manifold; iii) boundary conditions on the gauge functions λ associated to the softly broken gauge symmetries, *viz.*

$$\lambda|_{\partial M} \quad \text{for } \lambda \in \mathfrak{g}_1/\mathfrak{t}; \quad (3.50)$$

and iv) the boundary conditions (3.48) on the Lagrange multipliers.

3.2.3 Duality extended spectral flow with Lagrange multipliers

The equations of motion $\mathcal{R}^i \approx 0$ of the extended Lagrangian system $Z^i = (A, B; U, V)$ with bilinear P and Q structures (*i.e.* linear \mathcal{F} and $\widetilde{\mathcal{F}}$ functions) are compatible with the extended flow equations $L_1 A \approx 0 \approx L_1 B$ (or equivalently $L_1 B' \approx 0$) and

$$L_1 U \approx \mu_1 V' \star (\partial_{\nu} A) - \mu_2 (\partial_{\nu} A) \star V', \quad L_1 V' \approx \mu_1 V' \star (\partial_{\nu} B') + \mu_2 (\partial_{\nu} B') \star V', \quad (3.51)$$

with flow operator L_1 given by (2.19) and the redefinition

$$B = \nu \mathbf{1} + gB', \quad V = gV', \quad \nu, g \in \mathbb{C}. \quad (3.52)$$

We have not found any generalization of the spectral flow to the Lagrangian systems with higher-order P - or Q -structures (*i.e.* nonlinear \mathcal{F} or $\widetilde{\mathcal{F}}$ functions).

3.2.4 Consistent truncations off shell

Reality conditions can be imposed off shell by requiring the action to be either real or purely imaginary, *viz.*

$$(S_{\text{bulk}}^{\text{cl}})^\dagger = \epsilon_S S_{\text{bulk}}^{\text{cl}}, \quad (3.53)$$

leading to the following reality conditions on the Lagrange multipliers and the function $\widetilde{\mathcal{F}}$ appearing in the generalized P -structure:

$$(U_{[p]})^\dagger = \epsilon_S (\epsilon_{\mathbb{R}})^{n+\frac{p}{2}} U_{[p]}, \quad (V_{[p]})^\dagger = -\epsilon_S (\epsilon_{\mathbb{R}})^{n+\frac{p+1}{2}} V_{[p]}, \quad (3.54)$$

$$(\widetilde{\mathcal{F}}_0(\lambda))^\dagger = -\epsilon_{\mathbb{R}} \widetilde{\mathcal{F}}_0(\epsilon_S (\epsilon_{\mathbb{R}})^n \lambda^\dagger), \quad \left(\widetilde{\mathcal{F}}_I(\lambda)\right)^\dagger = \widetilde{\mathcal{F}}_{\bar{I}}(\epsilon_S (\epsilon_{\mathbb{R}})^n \lambda^\dagger), \quad (3.55)$$

$$\left(\widetilde{\mathcal{F}}_{I\bar{J}}(\lambda)\right)^\dagger = \epsilon_{\mathbb{R}} \widetilde{\mathcal{F}}_{\bar{I}I}(\epsilon_S (\epsilon_{\mathbb{R}})^n \lambda^\dagger). \quad (3.56)$$

From $\text{Tr}[\pi_k(\cdot)] = \text{Tr}[\cdot]$ it follows that in the case of π_k -projection then the k -parities must be correlated as follows:

$$\pi_k(U_{[p]}) = -\epsilon_k^{n+\frac{p}{2}} U_{[p]}, \quad \pi_k(V_{[p]}) = \epsilon_k^{n+\frac{p+1}{2}} V_{[p]}, \quad (3.57)$$

$$\widetilde{\mathcal{F}}_0(-(\epsilon_k)^n \lambda) = \epsilon_k \mathcal{V}_0(\lambda), \quad \widetilde{\mathcal{F}}_I(-(\epsilon_k)^n \lambda) = (-1)^{I+1} \widetilde{\mathcal{F}}_I(\lambda), \quad (3.58)$$

$$\widetilde{\mathcal{F}}_{I\bar{J}}(-(\epsilon_k)^n \lambda) = \epsilon_k (-1)^{I+\bar{J}} \widetilde{\mathcal{F}}_{I\bar{J}}(\lambda). \quad (3.59)$$

To fix spacetime parity one may impose $(\epsilon, \tilde{\epsilon} = \pm 1)$

$$P(U_{[p]}) = \epsilon (\epsilon \tilde{\epsilon})^{n+\frac{p}{2}} U_{[p]}, \quad P(V_{[p]}) = (\epsilon \tilde{\epsilon})^{n+\frac{p+1}{2}} V_{[p]}, \quad (3.60)$$

$$\widetilde{\mathcal{F}}_0(\epsilon (\epsilon \tilde{\epsilon})^n \lambda) = \epsilon \tilde{\epsilon} \mathcal{V}_0(\lambda), \quad \widetilde{\mathcal{F}}_{\bar{I}}(\lambda) = \mathcal{V}_{\bar{I}}(\epsilon (\epsilon \tilde{\epsilon})^n \lambda), \quad \widetilde{\mathcal{F}}_{I\bar{J}}(\lambda) = \epsilon \tilde{\epsilon} \widetilde{\mathcal{F}}_{\bar{J}I}(\epsilon (\epsilon \tilde{\epsilon})^n \lambda). \quad (3.61)$$

Finally, assuming $\text{Tr}[\tau(\cdot)] = \text{Tr}[\cdot]$, the projection to the minimal bosonic model takes the form

$$\tau(U_{[p]}) = (-1)^{n+\frac{p}{2}} U_{[p]}, \quad \tau(V_{[p]}) = (-1)^{n+\frac{p+1}{2}} V_{[p]}, \quad (3.62)$$

$$\widetilde{\mathcal{F}}_0((-1)^n \lambda) = \widetilde{\mathcal{F}}_0(\lambda), \quad \widetilde{\mathcal{F}}_I((-1)^n \lambda) = \widetilde{\mathcal{F}}_I(\lambda), \quad (3.63)$$

$$\widetilde{\mathcal{F}}_{I\bar{J}}((-1)^n \lambda) = \widetilde{\mathcal{F}}_{I\bar{J}}(\lambda). \quad (3.64)$$

3.3 Even-dimensional bulk ($p \in 2\mathbb{N} + 1$)

In the case of an even-dimensional bulk, say of dimension $\hat{p} + 1 = 2n$, one has the action

$$S_{\text{bulk}}^{\text{cl}}[A, B; S, T] = \int_M \text{Tr} \left[S \star DB + T \star (F + \mathcal{F}) + \mathcal{W}(S; J^I, J^{\bar{I}}, J^{I\bar{J}}) \star T \right], \quad (3.65)$$

where \mathcal{W} is an interaction \star -function obeying

$$\mathcal{W}(-\lambda) = \mathcal{W}(\lambda), \quad \mathcal{W}(0) = 0, \quad (3.66)$$

and the form degrees are assigned as follows:

$$A = \sum_{m=1,3,\dots,\hat{p}} A_{[m]}, \quad B = \sum_{m=0,2,\dots,\hat{p}-1} B_{[m]}, \quad (3.67)$$

$$S = \sum_{m=1,3,\dots,\hat{p}} S_{[m]}, \quad T = \sum_{m=0,2,\dots,\hat{p}-1} T_{[m]}. \quad (3.68)$$

The variational principle yields the generalized curvatures

$$\mathcal{R}^A = F + \mathcal{U} + \mathcal{W}(S), \quad \mathcal{R}^B = DB - (T\partial_S) \star \mathcal{W}(S), \quad (3.69)$$

$$\mathcal{R}^S = DS + (T\partial_B) \star \mathcal{F}, \quad \mathcal{R}^T = DT + [S, B]_\star. \quad (3.70)$$

The action is gauge invariant and the equations of motion are integrable in the case of

$$\text{bilinear } Q\text{-structure} : \mathcal{F} = J \star B, \quad (3.71)$$

for which the integrability of \mathcal{R}^T follows using the identity

$$\{S, (T\partial_S) \star \mathcal{W}\}_\star \equiv [T, \mathcal{W}]_\star, \quad (3.72)$$

that holds for general even \star -functions \mathcal{W} . The Cartan gauge transformations off shell are given by the on-shell transformations.

4 Discussions

Let us summarize our results, speculate on future directions and finally conclude by trying to place our work and ideas into the more general state of affairs.

4.1 Summary

In this paper we presented an action principle for a duality extended version of Vasiliev's equations for interacting higher spin gauge fields (including gravity) in four dimensions.

The duality extended version consists of differential forms of degrees $p \in \{0, 1, 2, \dots\}$ forming two master fields $B = B_{[0]} + B_{[2]} + \dots$ and $A = A_{[1]} + A_{[3]} + \dots$, and their Lagrange multipliers which are differential forms of dual form degrees of degrees $\hat{p} - p$ where $\hat{p} + 1$ is the dimension of the base manifold (including the twistor Z -space). The initial and boundary data associated with the Lagrange multipliers are removed by means of boundary conditions compatible with the variational principle. As a result, the Lagrange multipliers can be set equal to zero on shell, leaving A and B subject to the unfolded equations of motion $dA + A \star A + J \star B \approx 0$ and $dB + A \star B - B \star A \approx 0$ where $J = J_{[2]} + J_{[4]}$ is a closed and central element. This system contains Vasiliev's original equations in degrees zero and one, *viz.* $dA_{[1]} + A_{[1]} \star A_{[1]} + J_{[2]} \star B_{[0]} \approx 0$ and $dB_{[0]} + A_{[1]} \star B_{[0]} - B_{[0]} \star A_{[1]} \approx 0$.

An important point that remains to be established is whether the coupling $J_{[4]} \star B$ is nontrivial in the sense that it cannot be redefined away. In Vasiliev's original system, the coupling $J_{[2]}^2 \star B$ (and its hermitian conjugate) is nontrivial; it is indeed this term that reproduces the nontrivial interactions in the second order in curvature in the effective unfolded equations of motion in the perturbative expansion around a non-degenerate vierbein [37]. The reason $J_{[2]}^2 \star B$ is nontrivial is that the central term $J_{[2]}^2$ contains the inner Kleinian κ (that becomes a Dirac delta function in the Weyl order of the (Y, Z) -oscillator algebra). We note that also $J_{[4]}$ contains such "singular" elements, namely $J_{[4]}^{1\bar{2}} \star B$ (and its hermitian conjugate) and $J_{[4]}^{\bar{1}2} \star B$.

The duality-extended (A, B) -system is perturbatively equivalent to Vasiliev's original $(A_{[1]}, B_{[0]})$ -system:

- i) both systems share the same Weyl zero-form $B_{[0]}$; this master field contains the initial data associated to the Weyl curvature tensors, which contain one-particle states and other local deformations of the system such as for example the massive parameters of the black-hole solution of [38].
- ii) the master fields with positive form degree (including $A_{[1]}$) bring gauge functions on shell. In topologically broken phases, the boundary values of gauge functions associated with topologically broken gauge symmetries may contribute to observables; see Appendix A. Thus the original and duality-extended systems share the same observable gauge functions in the unbroken phase (where

no gauge functions are observable) and in broken phases where projections of $A_{[1]}$ are broken (such as for example the π -odd projection containing the ordinary vierbein).

We wish to stress, however, that if one has an exact solution to the duality-extended (A, B) -system, then it by construction contains an exact solution to the original system. As known from [39], there exist exact solutions of the original system for which the connections exhibit critical behaviors for finite amplitudes of $B_{[0]}$ (as can be described invariantly using zero-form invariants). Thus it is not clear whether a given exact solution to the original system can be uplifted to the duality-extended system, as new critical phenomena may arise in potentials in the duality-extended sector.

We also wish to stress that the action principle involves an integration over a base manifold given by the product of an ordinary commuting base manifold (containing four-dimensional spacetime) and the non-commutative twistor Z -space. The Lagrangian also contains an additional integration over the internal twistor Y -space — which one may think of as contracting indices related to various representations of an internal higher-spin Lie algebra.

In this sense, if one was to take our action principle seriously as a starting point for quantizing higher-spin gravity, one would have to address the issue of boundary conditions on the internal connection $(A_\alpha, A_{\dot{\alpha}})$ in Z -space. In the standard perturbative expansion in the Weyl zero-form $B_{[0]}$, it is usually assumed that $(A_\alpha, A_{\dot{\alpha}})$ is pure gauge in the limit where $B_{[0]}$ vanishes. However, as found in [39], there are “topologically nontrivial” exact solutions based on projectors in which $(A_\alpha, A_{\dot{\alpha}})$ remains nontrivial for vanishing $B_{[0]}$, whose physical meaning remains to be understood better.

4.2 Outlook: AKSZ-BV quantum action and unfolded quantum field theory

The action principle proposed in this paper is an example of a generalized Hamiltonian action principle for an associative free differential algebra on a noncommutative base manifold. More generally, as far as the off-shell formulation of free differential algebras is concerned, one may think of three different levels of complexity depending on whether the algebra is associative and commutative, or associative and non-commutative, or of strongly homotopy associative type. In the commutative case, the BV quantum action is of the AKSZ-BV type and it has been proposed that the perturbative quantization (with suitable boundary conditions on Lagrange multipliers) yields master theories of the homotopy type (with l -ary products arising via terms in the Hamiltonian that are of l -th order in the Lagrange multipliers).

In our case, there exists a quantum action of AKSZ-BV type which we shall present elsewhere.

Moreover, the classical $(A, B; U, V)$ system extends naturally to the strongly homotopy associative case and there are indications that its completion off shell leads to an AKSZ-BV-like quantum action (within a suitable Noether procedure). It is thus tempting to speculate that there exist quantum theories based on layers on “ n -quantized” unfolded quantum field theories such that each layer is the master theory of the layer below with radiative corrections interpreted as a topological sum, giving rise to third-quantization.

Pursuing these ideas, one is led to attempt to identify Vasiliev’s equations as the master equations for an underlying first-quantized topological open string: the system on the commutative manifold appears related to an underlying A-model [40]; and the system on the noncommutative twistor space appears related to a B-model [41, 40]. More generally, one may deform the bulk action with various topological vertex operators inserted on finite-dimensional sub-manifolds: these are gauge-invariant functionals whose variations vanish on shell (so that the standard first-order action is an example of such a deformation) and whose values on shell can be interpreted as amplitudes [14, 20, 21, 40]. There are many such deformations, each of which one may seek to relate to an underlying first-quantized dual, such as for example the holographic dual in three dimensions for which one may propose a topological vertex operator that is a four-form [42, 43].

The perturbations of the bulk action by various operators also provides a systematic approach to symmetry breaking mechanisms: for example, one has topological mechanisms (homotopy phases), spontaneous mechanisms (classical solutions) and dynamical mechanisms (radiative corrections); for a discussion, see for example [40].

More radically, one may go so far as to elevate the aforementioned layered structure of unfolded quantum field theories into a *quantum gauge principle*, i.e. a set of mathematical rules that are nontrivial in the sense that they are meant to hold for any *physical* (quantum) system. In particular, the Cartan integrable free differential algebra of the n th layer, with its exterior derivative d (on a base manifold) and Q -structure (in a target space), should arise from the BRST operator of the $(n - 1)$ -quantized system (subject to radiative corrections but with trivial topology as the topological sum of the $(n - 1)$ st layer should correspond to the radiative corrections of the n th layer). In other words, the quantum gauge principle is meant to contain Cartan’s version of Weyl’s classical gauge principle.

In other words, the idea is that generic quantum system should *not* abide by the quantum gauge principle making it nontrivial. We believe, however, that the Vasiliev system is a candidate for (a massless sector of) a system compatible with the quantum gauge principle [40].

4.3 Conclusions

As far as four-dimensional higher-spin gravities are concerned, the only fully nonlinear models that are known up to this date are those that have been obtained within Vasiliev's formalism. Vasiliev's formalism provides a general framework for higher-spin gravities based on free differential algebras on noncommutative manifolds taking their values in internal associative (super)algebras.

All models arising within this framework are based on one and the same universal equation of motion; different models arise by choosing different base manifolds and associative algebras. In this sense, all models arising within Vasiliev's framework can be viewed as various Yang–Mills and supersymmetric extensions of a basic minimal bosonic model consisting perturbatively of a scalar field, a metric and a tower of Fronsdal tensors of ranks $\{4, 6, \dots\}$.

Strictly speaking, these perturbative formulations arise only under a set of extra assumptions (on boundary conditions in twistor spaces); whether the resulting perturbative models exhaust all mathematical possibilities within the perturbative Fronsdal programme is an open problem though there are uniqueness theorems to low orders.

Remarkably, notwithstanding its somewhat peculiar features in comparison to the more traditional approach to lower-spin gravities, the perturbative expansions of Vasiliev's equations around its anti-de Sitter vacuum appear paradigmatic as far as holography is concerned, that is, it reproduces the simplest possible candidates for holographic duals of higher-spin gravities [44, 45, 46, 47]; see for example the recent works in [48, 49, 50].

Vasiliev's equations admit, however, exact solutions that involve moduli that are not visible in the perturbative Fronsdal Programme (for example solutions activating the internal connections in twistor space but not the Weyl tensors). The formalism also admits extensions by differential forms whose exterior derivatives vanish identically in the linearized approximation which one may think of as analogs of the three-dimensional gauge fields¹.

Taken altogether, the state of affairs motivates a more careful examination of whether the full field content of Vasiliev's unfolded formalism should be treated as the actual fundamental field content, and not only as part of a resourceful trick aimed at obtaining effective Fronsdal equations. In this approach, the aim becomes to include all unfolded variables (differential forms) into the action principle, which leads more or less directly to the type of generalized Hamiltonian bulk actions considered in this paper

¹These forms appear in the k -independent part of $B_{[0]}$ and the k -linear part of $A_{[1]}$.

and in fact already considered in [3], albeit in its simpler version without any Poisson structure.

These action principles lend themselves naturally to the BRST treatment leading to generalized AKSZ-BV models, which is the stage at which we are now. The resulting open problem is how to connect back to the perturbative quantization scheme within the Fronsdal Programme with its clear physical interpretation. To this end it is natural to examine various perturbations of the bulk action, which we leave for future studies.

Note added: The results in this paper were partly presented by P. S. at the IVth Sakharov International Conference, May 2009.

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A Free differential algebras on non-commutative base manifolds

Vasiliev's on-shell formulation of higher-spin gravity makes use of a version of unfolded dynamics that is based on associative free differential algebras with central and closed terms. Such an algebra encodes the following key structures:

$$(\mathfrak{B}, \mathfrak{A}, \star, d; \mathfrak{J}; \mathcal{I}, \overrightarrow{\mathcal{Q}}; \mathfrak{t}) ,$$

and it describes the moduli space $\mathcal{M}_{\mathfrak{t}}$ of \mathfrak{A} -valued sections $\{Z^i\}_{i \in \mathcal{I}}$ over a noncommutative base manifold \mathfrak{B} , subject to universally Cartan integrable flatness conditions on generalized curvatures

$$\mathcal{R}^i := dZ^i + \mathcal{Q}^i(Z, J) \approx 0 , \quad i \in \mathcal{I} , \quad (\text{A.1})$$

and defined modulo unbroken Cartan gauge transformations generated by \mathfrak{t} , a subalgebra of the Cartan gauge algebra \mathfrak{g} .

The $\{Z^i\}$ are the fundamental (classical) fields of the unfolded system; we refer to Z^i as the master field of flavor i . The master fields are differential forms in degrees $p_i \equiv \deg(Z^i) \in \mathbb{N}$ (including zero-forms). They can be acted upon with the exterior derivative d and composed using the associative

noncommutative product $\star \equiv \star \wedge$ combining the product on \mathfrak{A} and the composition of differential forms on \mathfrak{B} (represented by symbols). The following rules apply:

$$\deg(Z^i \star Z^j) = \deg(Z^i) + \deg(Z^j), \quad \deg(d) = 1, \quad (\text{A.2})$$

$$d(Z^i \star Z^j) - (dZ^i) \star Z^j - (-1)^{\deg(Z^i)} Z^i \star (dZ^j) \equiv 0, \quad (\text{A.3})$$

$$(Z^i \star Z^j) \star Z^k - Z^i \star (Z^j \star Z^k) \equiv 0. \quad (\text{A.4})$$

Locally, in the coordinate charts $\mathfrak{B}_\xi \subset \mathfrak{B}$, labelled here by an additional chart index, the sections have local representatives

$$Z_\xi^i \in \Omega(\mathfrak{B}_\xi) \otimes \mathfrak{A}. \quad (\text{A.5})$$

The structure functions $\mathcal{Q}^i(Z, J)$ in (A.1) are given by \star -power expansions in Z^i and an additional set $\{J^I\}$ of globally defined elements that are central and closed, *viz.*

$$J^I \in \Omega(\mathfrak{B}) \otimes \mathfrak{A}, \quad dJ^I \equiv 0, \quad J^I \star Z^i - Z^i \star J^I \equiv 0, \quad (\text{A.6})$$

hence generating a closed and central subalgebra

$$\mathfrak{J} \subset \Omega(\mathfrak{B}) \otimes \mathfrak{A}. \quad (\text{A.7})$$

The structure functions can thus be presented as

$$\mathcal{Q}^i = \sum_n \mathcal{Q}_{j_1, \dots, j_n}^i(J^I) \star Z^{j_1} \star \dots \star Z^{j_n} \quad (\text{A.8})$$

with coefficients $\mathcal{Q}_{j_1, \dots, j_n}^i(J^I) \in \mathfrak{J}$ that need not be graded symmetric in their lower flavor indices (due to the non-commutativity of \star). The universal Cartan integrability of (A.1) is tantamount to compatibility with $d^2 \equiv 0$ on base manifolds \mathfrak{B} of arbitrary dimension. Using the notation for \star -vector fields (see Appendix C), this amounts to that

$$\overrightarrow{\mathcal{Q}} \star \overrightarrow{\mathcal{Q}} \equiv 0, \quad \overrightarrow{\mathcal{Q}} := \mathcal{Q}^i(Z^j, J^I) \partial_i, \quad (\text{A.9})$$

or equivalently, that the coefficients obey

$$\sum_{n_1+n_2=n-1} \sum_{m=1}^{n_1} \mathcal{Q}_{j_1, \dots, j_{m-1}, k, j_m, \dots, j_{n_1-1}}^i(J^I) \star \mathcal{Q}_{j_{n_1}, \dots, j_n}^k(J^I) \equiv 0, \quad (\text{A.10})$$

where the flavor indices j_1, \dots, j_n are not subject to any graded symmetry.

The universal Cartan integrability implies that the constraint surface remains invariant under the Cartan gauge transformations

$$\delta_\varepsilon Z^i \equiv \overrightarrow{\mathcal{F}}_{\varepsilon, Z} \star Z^i := d\varepsilon^i - \overrightarrow{\varepsilon} \star \mathcal{Q}^i, \quad \overrightarrow{\varepsilon} := \varepsilon^i \partial_i, \quad (\text{A.11})$$

which are linear in gauge parameters ε^i and in general nonlinear in Z^i . These transformations form a soft gauge algebra \mathfrak{g} that exponentiates into generalized (or soft) group elements

$$\mathcal{G}_{\lambda, Z} := \exp_\star \overrightarrow{\mathcal{F}}_{\lambda, Z} \quad (\text{A.12})$$

generated by (finite) gauge functions λ^i . The space \mathcal{M}_ξ of locally defined solutions to $\mathcal{R}_\xi^i := dZ_\xi^i + \mathcal{Q}^i(Z_\xi^j, J^I) \approx 0$ is given formally by Cartan gauge orbits, *viz.*

$$\mathcal{M}_\xi = \{ \mathcal{G}_{\lambda, Z} \star Z^i : \lambda = \lambda_\xi, Z^i = Z_{C_\xi}^i \}, \quad (\text{A.13})$$

where λ_ξ^i and $Z_{C_\xi}^i$ are locally defined gauge functions and reference solutions, respectively; the reference solution obeys i) the constraints $dZ_{C_\xi}^i + \mathcal{Q}^i(Z_{C_\xi}, J) \approx 0$; ii) an initial datum $(Z_{C_\xi}^i|_{[0]})|_{p_\xi} = C_\xi^i$ where $p_\xi \in \mathfrak{B}_\xi$ is a base point and $(\cdot)|_{[0]}$ denotes the projection to zero form degree; and iii) a physical gauge condition (to select a well-defined particular solution and avoid over-representation). Interestingly enough, the unfolded formulation of higher-spin gravities appears amenable to the implementation of the above form of Cartan integrability at least in sub-sectors of the theory.

The moduli space \mathcal{M}_t is obtained by first gluing together locally defined modules \mathcal{M}_ξ by means of transition functions valued in the unbroken gauge algebra $\mathfrak{t} \subseteq \mathfrak{g}$, *viz.*

$$\mathcal{M}_\xi \cong \mathcal{G}_\xi^{\xi'} \star \mathcal{M}_{\xi'}, \quad \mathcal{G}_\xi^{\xi'} := \exp_\star \overrightarrow{\mathcal{F}}_{\lambda_\xi^{\xi'}, Z_{\xi'}^i}, \quad \lambda_\xi^{\xi'} \in \mathfrak{t}, \quad (\text{A.14})$$

where the parameters are defined on (cylinders homotopic to) the overlaps $\mathfrak{B}_\xi \cap \mathfrak{B}_{\xi'}$ (we are assuming that $\mathfrak{B} = \bigcup_\xi \mathfrak{B}_\xi$). The gluing compatibility implies that

$$Z_{C_\xi}^i = Z_C^i \quad \text{for all } \xi, \quad (\text{A.15})$$

where thus C is (gauge non-invariant) constant of motion, and that

$$\mathcal{G}_\xi^{\xi'} = \mathcal{G}_\xi \star (\mathcal{G}_{\xi'})^{-1}, \quad (\text{A.16})$$

which is a nontrivial gluing condition on the gauge functions. The coordinates on \mathcal{M}_t are gauge-invariant and intrinsically defined observables \mathcal{O}_t , that is, functionals of the master fields constructed out of local

functionals that are manifestly \mathfrak{t} -invariant off shell and intrinsically defined, *i.e.* independent under any particular choices of local data on the base manifold and hence manifestly diffeomorphism invariant (consequently non-local). The manifest \mathfrak{t} -invariance implies that $\mathcal{G}_\xi^{\xi'} \sim \mathcal{U}_\xi \star \mathcal{G}_\xi^{\xi'} \star (\mathcal{U}_{\xi'})^{-1}$ where \mathcal{U}_ξ is generated by \mathfrak{t} . Thus, in view of (A.16), one has that

$$\mathcal{G}_\xi \sim \mathcal{U}_\xi \star \mathcal{G}_\xi \quad \text{where } \mathcal{U}_\xi \text{ is generated by } \mathfrak{t}, \quad (\text{A.17})$$

that is, the gauge functions in $\mathcal{M}_\mathfrak{t}$ can be taken to be valued in the coset $\mathfrak{g}/\mathfrak{t}$.

For example, one may consider homotopy charges given by integrals

$$\mathcal{O} := \oint_{\Sigma'} (\omega^R + k^R), \quad \Sigma' \in [\Sigma] \quad (\text{A.18})$$

over nontrivial p_R -cycles $[\Sigma]$ of p_R -forms $\omega^R[Z, J]$ and $k^R[Z, J]$ that are manifestly \mathfrak{t} -invariant, *i.e.*

$$\delta_\varepsilon(\omega^R, k^R) \equiv 0, \quad \varepsilon \in \mathfrak{t}, \quad (\text{A.19})$$

and defined by the equivariant cohomology system

$$d\omega^R + f^R(\omega) \approx 0, \quad f^R(\omega)|_{\Sigma_{\text{cyl}}} \approx dk^R|_{\Sigma_{\text{cyl}}}, \quad (\text{A.20})$$

where Σ_{cyl} is a cylinder of finite thickness containing Σ ; the homotopy invariance of de Rham cohomology classes then implies that $H^{p_R+1}(\Sigma_{\text{cyl}}) = 0$ so that $f^R|_{\Sigma_{\text{cyl}}}$ must be exact, that is, given by the exterior derivative of some p_R -form k^R that is globally defined on Σ (and hence gauge invariant). Thus the integral over Σ , which must necessarily be split into several charts, say $\{\Sigma_\xi\}$, makes sense and is independent of the choice of Σ' . A variation $\delta_\varepsilon \lambda^i = \varepsilon^i$ in the gauge functions thus induces a change in $(\omega^R + k^R)|_{\Sigma_\xi}$ given by

$$\delta_\varepsilon(\omega^R + k^R)|_{\Sigma_\xi} = dX_\xi(\varepsilon_\xi), \quad (\text{A.21})$$

where $X_\xi(\varepsilon_\xi)$ is a linear functional in ε_ξ^i . By the \mathfrak{t} -invariance, one has that $X_\xi(\varepsilon_\xi)$ is invariant under \mathfrak{t} -transformations that act simultaneously on Z^i and the gauge parameter (*c.f.* the BRST treatment where the gauge parameter is promoted into a ghost). It follows that

$$\delta_\varepsilon \mathcal{O}_\mathfrak{t} = \sum_\xi \oint_{\partial \Sigma_\xi} X_\xi(\varepsilon_\xi), \quad (\text{A.22})$$

which can be split into contributions from chart boundaries in the interior of \mathfrak{B} and from true boundaries of \mathfrak{B} . The former must cancel identically if one assumes that the choice of where to cut the interior of \mathfrak{B}

into charts should not be of no importance. Taking into account the signs coming from orientation, this is a consequence of the fact that $\{\lambda^i\}$ forms a globally defined section (of the soft t-bundle) as stated in (A.16). One thus has

$$\delta_\varepsilon \mathcal{O}_t = \sum_\xi \oint_{\partial \mathfrak{B} \cap \partial \Sigma_\xi} X_\xi(\varepsilon_\xi) , \quad (\text{A.23})$$

that is the only physical dependence of the gauge functions enters via their boundary values, which one may view as an unfolded version of the holographic principle.

B Duality extension

We consider an associative free differential algebra consisting of master fields Z^i and structure coefficients $\mathcal{Q}_{j_1, \dots, j_n}^i(J^I)$ of fixed degrees, say $\deg(Z^i) = p_i \in \mathbb{N}$ and $\deg(\mathcal{Q}_{j_1, \dots, j_n}^i) = p_{j_1 \dots j_n}^i \in 2\mathbb{N}$. This system can always be duality extended (without adding any new local degrees of freedom) by i) replacing Z^i by $\widehat{Z}^i := \sum_k Z_{[p_i+2k]}^i$; and ii) exploiting field redefinitions to introduce coupling constants $g_{[0]}$ and then replace these by $\widehat{g}(J^I) := \sum_k g_{[2k]}$. It follows that the extended system $\{\widehat{Z}^i, \widehat{g}\}$ contains the original system $\{Z_{[p_i]}^i, g_{[0]}\}$ as a consistent subsystem, though the added master fields $Z_{[p_i+2k]}^i$ with $k > 0$ cannot in general be set equal to zero, since they are sourced from $\{Z_{[p_i]}^i\}$ via terms involving the new couplings $g_{[2k]}$ with $k > 0$.

One may refer to the duality extension as non-trivial if the central elements cannot be removed by redefining the master fields; we are not aware of any general condition that guarantees non-triviality.

C Further details: \star -vector fields and Cartan integrability

In this Appendix we go into the technical details of \star -functions, \star -vector fields and Cartan integrability that were introduced in Appendix A. Let us first recall the general idea of a free differential algebra on a non-commutative base manifold \mathfrak{B} consisting of graded associative algebras \mathfrak{R}_ξ generated by sets $\{Z_\xi^i\}_{i \in \mathcal{I}}$ of locally defined differential forms subject to generalized curvature constraints

$$\mathcal{R}_\xi^i := dZ_\xi^i + \mathcal{Q}^i(Z_\xi, J) \approx 0 , \quad (\text{C.1})$$

where $\overrightarrow{\mathcal{Q}} := \mathcal{Q}^i \partial_i$ is a composite \star -vector field of total degree one subject to the Cartan integrability condition

$$\overrightarrow{\mathcal{Q}} \star \mathcal{Q}^i \equiv 0 . \quad (\text{C.2})$$

Here we use the following notation and conventions (see [42, 40, 43] for further details):

- (i) ξ labels charts $\mathfrak{B}_\xi \subset \mathfrak{B}$ with coordinates Ξ_ξ^M of degree zero and differentials $d\Xi_\xi^M$ of degree one generating \mathbb{N} -graded associative \star -product algebras

$$\Omega_\xi \equiv \text{Env}[\Xi_\xi^I, d\Xi_\xi^I] \quad (\text{C.3})$$

modulo the graded \star -commutators

$$[\Xi_\xi^M, \Xi_\xi^N]_\star = 2i\Pi^{MN}, \quad [\Xi_\xi^M, d\Xi_\xi^N]_\star = 0, \quad [d\Xi_\xi^M, d\Xi_\xi^N]_\star = 0, \quad (\text{C.4})$$

where Π^{MN} is a constant matrix (defining a canonical Poisson structure $\Pi = \Pi^{MN}\partial_M \otimes \partial_N$);

- (ii) the action of the exterior derivative $d = d\Xi_\xi^M \partial / \partial \Xi_\xi^M$ in Ω_ξ is defined by declaring that

$$d(\Xi_\xi^M) = d\Xi_\xi^M, \quad d(f \star g) = (df) \star g + (-1)^{\deg f} f \star (dg), \quad (\text{C.5})$$

for elements $f, g \in \Omega$ such that f has fixed form degree $\deg f$; one has

$$d^2 \equiv 0. \quad (\text{C.6})$$

- (iii) the locally defined differential forms $Z_\xi^i \in \Omega_\xi^{[p_i]} \otimes \Theta^i$, where $\Omega_\xi^{[p_i]}$ is the subspace of Ω_ξ of fixed form degree m_i and Θ^i can be either are finite-dimensional internal tensors (such as for example Lorentz tensors) or sectors of an internal associative algebra \mathfrak{A} ;

- (iv) the graded associative \star -product algebra $\mathfrak{R}_\xi := \text{Env}[Z_\xi^i] \otimes \mathfrak{J}$ where \mathfrak{J} is a space of central and d -closed elements (including the identity), *i.e.* if $\mathcal{F}(Z_\xi^i) \in \mathfrak{R}_\xi$ then

$$\mathcal{F} = \sum_{n \geq 0} \mathcal{F}_{j_1 \dots j_n} \star Z_\xi^{j_1} \star \dots \star Z_\xi^{j_n}, \quad \mathcal{F}_{j_1 \dots j_n} \in \mathfrak{J}; \quad (\text{C.7})$$

- (v) a composite \star -vector field $\overrightarrow{\mathcal{X}}$ is a graded inner derivation of \mathfrak{R} , *i.e.* if $\mathcal{F}, \mathcal{F}' \in \mathfrak{R}$ then

$$\overrightarrow{\mathcal{X}} \star (\mathcal{F} \star \mathcal{F}') = (\overrightarrow{\mathcal{X}} \star \mathcal{F}) \star \mathcal{F}' + (-1)^{\deg(\overrightarrow{\mathcal{X}})\deg(\mathcal{F})} \mathcal{F} \star (\overrightarrow{\mathcal{X}} \star \mathcal{F}'), \quad (\text{C.8})$$

provided that $\overrightarrow{\mathcal{X}}$ and \mathcal{F} have fixed degrees. In components, one writes $\overrightarrow{\mathcal{X}} := \mathcal{X}^i(Z^j)\partial_i$ where $\mathcal{X}^i := \mathcal{X} \star Z^i$ (and $\partial_i \equiv \overrightarrow{\partial}_i$). The graded bracket between two composite \star -vector fields is defined by

$$[\overrightarrow{\mathcal{X}}, \overrightarrow{\mathcal{X}'}]_\star \star \mathcal{F} := \overrightarrow{\mathcal{X}} \star (\overrightarrow{\mathcal{X}'} \star \mathcal{F}) - (-1)^{\deg(\overrightarrow{\mathcal{X}})\deg(\overrightarrow{\mathcal{X}'})} \overrightarrow{\mathcal{X}'} \star (\overrightarrow{\mathcal{X}} \star \mathcal{F}), \quad (\text{C.9})$$

is a degree-preserving graded Lie bracket, *i.e.* $[\vec{\mathcal{X}}, \vec{\mathcal{X}}']_\star$ is a graded inner derivation obeying the graded Jacobi identity $[[\vec{\mathcal{X}}, \vec{\mathcal{X}}']_\star, \vec{\mathcal{X}}'']_\star + \text{graded cyclic} \equiv 0$. In components, one has

$$[\vec{\mathcal{X}}, \vec{\mathcal{X}}']_\star = \left(\vec{\mathcal{X}} \star \mathcal{X}^n - (-1)^{\deg(\mathcal{X})\deg(\vec{\mathcal{X}}')} \vec{\mathcal{X}}' \star \mathcal{X}^i \right) \partial_i. \quad (\text{C.10})$$

The Cartan integrability condition (C.2), that can be rewritten $[\vec{\mathcal{D}}, \vec{\mathcal{D}}]_\star \equiv 0$, amounts to that $\vec{\mathcal{D}}$ is a nilpotent composite \star -vector field of degree one. This condition ensures that the generalized curvature constraints $\mathcal{R}^i \approx 0$ are compatible with $d^2 \equiv 0$ without further algebraic constraints on the generating elements Z_ξ^i . One can also show that the nilpotency of $\vec{\mathcal{D}}$ is separately equivalent to that the generalized curvatures \mathcal{R}^i obey the generalized Bianchi identities

$$d\mathcal{R}^i - \vec{\mathcal{R}} \star \mathcal{Q}^i \equiv 0, \quad \text{where} \quad \vec{\mathcal{R}} := \mathcal{R}^i \partial_i, \quad (\text{C.11})$$

and transform into each other under the following Cartan gauge transformations

$$\delta_\varepsilon Z^i \equiv \mathcal{T}_\varepsilon^i := d\varepsilon^i - \vec{\varepsilon} \star \mathcal{Q}^i, \quad \text{where} \quad \vec{\varepsilon} := \varepsilon^i \partial_i \quad (\text{C.12})$$

and where ε^i is an element in $\Omega \otimes \Theta^i$ that is considered infinitesimal and independent of Z^i , *viz.*

$$\delta_\varepsilon \mathcal{R}^i = -\vec{\mathcal{R}} \star ((\vec{\varepsilon} \star \mathcal{Q}^i)). \quad (\text{C.13})$$

The closure relation reads

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] Z^i = \delta_{\varepsilon_{12}} Z^i - \vec{\mathcal{R}} \star \varepsilon_{12}^i, \quad (\text{C.14})$$

where the combined parameters ε_{12}^i 's are given by

$$\varepsilon_{12}^i = -\frac{1}{2} [\vec{\varepsilon}_1, \vec{\varepsilon}_2]_\star \star \mathcal{Q}^i. \quad (\text{C.15})$$

The above results can easily be obtained upon introducing the even \star -vector field

$$\vec{\mathcal{V}}_\varepsilon := (\vec{\varepsilon} \star \mathcal{Q}^i) \partial_i \quad (\text{C.16})$$

and using the following set of identities which are consequences of the first one:

$$[\vec{\mathcal{D}}, \vec{\mathcal{D}}]_\star \equiv 0, \quad [\vec{\mathcal{D}}, \vec{\mathcal{V}}_\varepsilon]_\star \equiv 0, \quad [\vec{\mathcal{V}}_{\varepsilon_1}, \vec{\mathcal{V}}_{\varepsilon_2}]_\star \equiv [\vec{\mathcal{D}}, \vec{\varepsilon}_{12}]_\star, \quad (\text{C.17})$$

where we recall that all the commutators are graded-commutators.

As discussed above, the local representatives \mathfrak{R}_ξ are glued together on overlaps $\mathfrak{B}_\xi \cup \mathfrak{B}_{\xi'}$ by means of the transitions $Z_\xi^i = \mathcal{G}_\xi^{\xi'} \star Z_{\xi'}^i$, where the transition functions $\mathcal{G}_\xi^{\xi'}$ are soft group elements given by

\star -exponentials of the Cartan gauge transformations as in (A.12). From the Leibnitz' rule (C.8) it follows that these transitions are indeed isomorphisms, *viz.*

$$\mathcal{G} \star \mathcal{F}(Z) = \mathcal{F}(\mathcal{G} \star Z), \quad \mathcal{G} \star (\mathcal{F} \star \mathcal{F}') = (\mathcal{G} \star \mathcal{F}) \star (\mathcal{G} \star \mathcal{F}'). \quad (\text{C.18})$$

We would like to show that, if Z^i satisfies the star-product equation $dZ^i + \mathcal{Q}^i(Z^j) \approx 0$, then $Z_\lambda^i := (\exp_\star[\vec{\mathcal{T}}_{\lambda,Z}]) \star Z^i$ where $\vec{\mathcal{T}}_\lambda := d\lambda^i \partial_i - \vec{\mathcal{V}}_\lambda$ [see (C.16)] satisfies the equation $dZ_\lambda^i + \mathcal{Q}^i(Z_\lambda, J) \approx 0$, thereby exhibiting the fundamental integrability of the unfolded equations in the case where the free differential algebra \mathcal{A} is endowed with a non-commutative star-product. We recall that

Lemma: The following commutation relation is true: $[\vec{\mathcal{T}}_\lambda, d]_\star \approx 0$, where the weak equality means an equality on the surface $\Sigma \equiv \{dZ^i + \mathcal{Q}^i(Z, J)\} = 0$.

Proof of the Lemma: On the surface Σ , the exterior derivative $d \approx \vec{\mathcal{Q}} - \vec{\Lambda}$, where

$$\vec{\Lambda} := d\lambda^i \frac{\partial}{\partial \lambda^i}. \quad (\text{C.19})$$

The proof is tantamount to showing that $[\vec{\mathcal{T}}_\lambda, \vec{\mathcal{Q}} - \vec{\Lambda}]_\star \star Z^i = 0 = [\vec{\mathcal{T}}_\lambda, \vec{\mathcal{Q}} - \vec{\Lambda}]_\star \star \lambda^i$ because then, using the fact that $[\vec{\mathcal{T}}_\lambda, \vec{\mathcal{Q}} - \vec{\Lambda}]_\star$ is a \star -vector field, it follows that $[\vec{\mathcal{T}}_\lambda, \vec{\mathcal{Q}} - \vec{\Lambda}]_\star \star \mathcal{F}(Z, \lambda) = 0$ for an arbitrary star-product function $\mathcal{F}(Z, \lambda)$.

(a) First of all, it is trivial to see that $[\vec{\mathcal{T}}_\lambda, \vec{\mathcal{Q}} - \vec{\Lambda}]_\star \star \lambda^i = 0$. Indeed, it gives $\vec{\mathcal{T}}_\lambda(d\lambda^i)$ which vanishes².

(b) That $[\vec{\mathcal{T}}_\lambda, \vec{\mathcal{Q}} - \vec{\Lambda}]_\star \star Z^i = 0$ is more difficult to show. For that, we write

$$\mathcal{Q}^i = \sum_n \mathcal{Q}_{j_1 \dots j_n}^i(J) \star Z^{j_1} \star \dots \star Z^{j_n}$$

where $\mathcal{Q}_{j_1 \dots j_n}^i \in \mathfrak{J}$ and compute

$$\begin{aligned} \vec{\mathcal{Q}} \star (\mathcal{T}_\lambda \star Z^i) &= - \sum_n \sum_{\beta < \alpha = 1}^n (-1)^{j_\beta + 1 + \dots + j_{\alpha-1}} \mathcal{Q}_{j_1 \dots j_n}^i \star Z^{j_1} \star \dots \star \mathcal{Q}^{j_\beta} \star \dots \star \lambda^{j_\alpha} \star \dots \star Z^{j_n} \\ &\quad - \sum_n \sum_{\alpha < \beta = 1}^n (-1)^{1 + j_\alpha + \dots + j_\beta} \mathcal{Q}_{j_1 \dots j_n}^i \star Z^{j_1} \star \dots \star \lambda^{j_i} \star \dots \star \mathcal{Q}^{j_\beta} \star \dots \star Z^{j_n}, \\ \vec{\mathcal{T}}_\lambda \star (\mathcal{Q} \star Z^i) &= [d\lambda^k - (\lambda^j \partial_j) \star \mathcal{Q}^k] \partial_k \star \mathcal{Q}^i, \\ \vec{\Lambda} \star (\mathcal{T}_\lambda \star Z^i) &= - \sum_n \sum_{\alpha=1}^n \mathcal{Q}_{j_1 \dots j_n}^i \star Z^{j_1} \star \dots \star d\lambda^{j_\alpha} \star \dots \star Z^{j_n}, \quad \mathcal{T}_\lambda \star (\Lambda \star Z^i) = 0. \end{aligned}$$

²We consider the algebra where the fields $\{Z^i\}$ and $\{\lambda^i\}$ are considered as independent, in accordance with the *BRST* treatment of gauge systems.

Regrouping all the terms, we find

$$[\vec{\mathcal{T}}_\lambda, \vec{\mathcal{D}} - \vec{\Lambda}]_\star \star Z^i = \mathcal{Q}^j \partial_j \star [(\lambda^k \partial_k) \star \mathcal{Q}^i] - [(\lambda^k \partial_k) \star \mathcal{Q}^j] \partial_j \star \mathcal{Q}^i \quad (\text{C.20})$$

which vanishes identically due to the second identity of (C.17).

Therefore, since $[\vec{\mathcal{T}}_\lambda, \vec{\mathcal{D}} - \vec{\Lambda}]_\star$ is a star-product vector field, it follows that $[\vec{\mathcal{T}}_\lambda, \vec{\mathcal{D}} - \vec{\Lambda}]_\star \star \mathcal{F}(Z, \lambda) = 0$ for an arbitrary star-product function $\mathcal{F}(Z, \lambda)$. \square

Using the above Lemma, we have that $Z_\lambda^i := (\exp_\star[\vec{\mathcal{T}}_\lambda]) \star Z^i$ satisfies the equation $dZ_\lambda^i + \mathcal{Q}^i(Z_\lambda^j) \approx 0$, since $dZ_\lambda^i \equiv d[(\exp_\star[\vec{\mathcal{T}}_\lambda]) \star Z^i] = (\exp_\star[\vec{\mathcal{T}}_\lambda]) \star dZ^i \approx -(\exp_\star[\vec{\mathcal{T}}_\lambda]) \star \mathcal{Q}^i(Z) \equiv -\mathcal{Q}^i(Z_\lambda)$. This proves the formal Cartan integrability of the star-product unfolded equations.

D The Vasiliev equations

In the case of Vasiliev's equations, the master fields are locally defined operators of the form

$$O_\xi(X_\xi^M, P_M^\xi, dX_\xi^M, dP_M^\xi; Z^\alpha, dZ^\alpha; Y^\alpha; e^i), \quad (\text{D.1})$$

where the non-vanishing commutators among the coordinates are

$$[X^M, P_N] = i\delta_N^M, \quad [Y^\alpha, Y^\beta]_\star = 2iC^{\alpha\beta}, \quad [Z^\alpha, Z^\beta]_\star = -2iC^{\alpha\beta}, \quad (\text{D.2})$$

with charge conjugation matrix³ $C^{\alpha\beta} = \epsilon^{\alpha\beta}$ and $C^{\dot{\alpha}\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}}$. The operators are represented by symbols $f[O_\xi]$ obtained by going to specific bases for the operator algebra which one may also think of as ordering prescriptions⁴. One may think of the symbols as functions $f(X, P, Z; Y; dX, dP, dZ)$ (with variables composed using commutative juxtaposition) on a correspondence space

$$\mathfrak{C} = \bigcup_\xi \mathfrak{C}_\xi, \quad \mathfrak{C}_\xi = \mathfrak{B}_\xi \times \mathfrak{Y}, \quad \mathfrak{B}_\xi = \mathfrak{M}_\xi \times \mathfrak{Z}. \quad (\text{D.3})$$

³We raise and lower quartet and doublet indices using the conventions $\Lambda^\alpha = C^{\alpha\beta} \Lambda_\beta$, and $\lambda^\alpha = \epsilon^{\alpha\beta} \lambda_\beta$ and $\lambda_\alpha = \lambda^\beta \epsilon_{\beta\alpha}$, and we use the notation $\Lambda \cdot \Lambda' = \Lambda^\alpha \Lambda'_\alpha$, and $\lambda \cdot \lambda' = \lambda^\alpha \lambda'_\alpha$ and $\bar{\lambda} \cdot \bar{\lambda}' = \bar{\lambda}^{\dot{\alpha}} \bar{\lambda}'_{\dot{\alpha}}$.

⁴The symbols are thus defined modulo similarity transformations generated by inner automorphisms (related to the higher-spin gauge transformations) as well as changes of the order prescription, that is, changes of basis of the operator algebra. These types of transformations may have a drastic effect on the mathematical nature of the symbols, that may change from being a smooth or real analytic into being singular or even distributions. Thus, in order to extract physically meaningful information from the master fields, one needs to develop the notion of observables \mathcal{O} , namely functionals of the locally defined master fields that are invariant under both gauge transformations and re-orderings. The construction of such functionals introduces various geometric concepts into the theory, such as flat connections, covariantly constant sections (going into decorated Wilson loops), equivariantly closed forms (used to define homotopy charges) and metrics (that yield minimal areas of closed cycles).

Working within a restricted class of orderings, referred to as universal orderings, the exterior derivative on \mathfrak{B} is given by

$$d = dX^M \partial_M + dP_M \partial^M + q, \quad q := dZ^\alpha \partial_\alpha. \quad (\text{D.4})$$

The master fields of the (duality-unextended) minimal bosonic model are an adjoint one-form

$$A = W + V, \quad (\text{D.5})$$

$$W = dX^M W_M(X, P, Z; Y) + dP_M W^M(X, P, Z; Y), \quad V = dZ^\alpha V_\alpha(X, P, Z; Y), \quad (\text{D.6})$$

and a twisted-adjoint zero-form

$$\Phi = \Phi(X, P, Z; Y); \quad (\text{D.7})$$

these fields obey the following projection and reality conditions⁵:

$$\tau(A, \Phi) = (-A, \pi(\Phi)), \quad (A, \Phi)^\dagger = (-A, \pi(\Phi)), \quad (\text{D.8})$$

where the maps τ , π , $\bar{\pi}$ and \dagger are defined by $d \circ (\tau, \pi, \bar{\pi}, \dagger) = (\tau, \pi, \bar{\pi}, \dagger) \circ d$ and⁶

$$\pi(y_\alpha, \bar{y}_{\dot{\alpha}}; z_\alpha, \bar{z}_{\dot{\alpha}}) = (-y_\alpha, \bar{y}_{\dot{\alpha}}; -z_\alpha, \bar{z}_{\dot{\alpha}}), \quad \pi(f \star g) = \pi(f) \star \pi(g), \quad (\text{D.9})$$

$$\bar{\pi}(y_\alpha, \bar{y}_{\dot{\alpha}}; z_\alpha, \bar{z}_{\dot{\alpha}}) = (y_\alpha, -\bar{y}_{\dot{\alpha}}; z_\alpha, -\bar{z}_{\dot{\alpha}}), \quad \bar{\pi}(f \star g) = \bar{\pi}(f) \star \bar{\pi}(g), \quad (\text{D.10})$$

$$\tau(y_\alpha, \bar{y}_{\dot{\alpha}}; z_\alpha, \bar{z}_{\dot{\alpha}}) = (iy_\alpha, i\bar{y}_{\dot{\alpha}}; -iz_\alpha, -i\bar{z}_{\dot{\alpha}}), \quad \tau(f \star g) = (-1)^{fg} \tau(g) \star \tau(f), \quad (\text{D.11})$$

$$(y_\alpha, \bar{y}_{\dot{\alpha}}; z_\alpha, \bar{z}_{\dot{\alpha}})^\dagger = (\bar{y}_{\dot{\alpha}}, y_\alpha; \bar{z}_{\dot{\alpha}}, z_\alpha), \quad (f \star g)^\dagger = (-1)^{fg} g^\dagger \star f^\dagger. \quad (\text{D.12})$$

The τ -projection removes all terms that are associated with the unfolded description of spacetime fermions as well as spacetime bosons with odd spin.

The full equations of motion of the minimal bosonic model with fixed interaction ambiguity amount to the statement that the full curvature $F = dA + A \star A$ is proportional to F , viz. $F + F \star J = 0$, via a deformed symplectic two-form J defined globally on correspondence space, obeying $\tau(J) = J^\dagger = -J$ and

$$dJ = 0, \quad [J, f]_\pi = 0, \quad (\text{D.13})$$

⁵Here we are focusing on the models containing spacetimes with Lorentzian signature and negative cosmological constant; for other signatures and signs of the cosmological constant, see [39].

⁶The rule $(f \star g)^\dagger = g^\dagger \star f^\dagger$ holds for both real and chiral integration domain.

for any f obeying $\pi\bar{\pi}(f) = f$, and where we have defined $[f, g]_\pi = f \star g - g \star \pi(f)$. In the minimal model,

$$J = -\frac{i}{4}(bdz^2 \kappa + \bar{b}d\bar{z}^2 \bar{\kappa}), \quad (\text{D.14})$$

where the chiral Klein operators⁷ are given in the normal-ordering by

$$\kappa = \exp(iy^\alpha z_\alpha), \quad \bar{\kappa} = \kappa^\dagger = \exp(-i\bar{y}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}}). \quad (\text{D.15})$$

By making use of field redefinitions $\Phi \rightarrow \lambda F$ with $\lambda \in \mathbb{R}$, $\lambda \neq 0$, the parameter b in J can be taken to obey

$$|b| = 1, \quad \arg(b) \in [0, \pi]. \quad (\text{D.16})$$

The phase breaks parity except in the following two cases:

$$\text{Type A model (parity-even physical scalar)} : b = 1, \quad (\text{D.17})$$

$$\text{Type B model (parity-odd physical scalar)} : b = i. \quad (\text{D.18})$$

The integrability of $F + F \star J = 0$ implies that $D F \star J = 0$, that is, $D F = 0$, where the twisted-adjoint covariant derivative $D F = F + A \star F - F \star \pi(A)$. This constraints is integrable since

$$D^2 F = F \star F - F \star \pi(F) = -F \star J \star F + F \star \pi(F) \star J = 0, \quad (\text{D.19})$$

using the constraint on F and (D.13).

Thus, in summary, the unfolded system describing the minimal higher-spin gravity is given by

$$F + \Phi \star J = 0, \quad D \Phi = 0, \quad dJ = 0, \quad (\text{D.20})$$

$$F = dA + A \star A, \quad D F = F + [A, F]_\pi, \quad (\text{D.21})$$

and the kinematic constraints D.8 which imply $[A, J]_\pi = [\Phi, J]_\pi = 0$.

⁷The quantities κ and $\bar{\kappa}$ are the Klein operators of the chiral Heisenberg algebras generated by (y_α, z_α) and $(\bar{y}_{\dot{\alpha}}, \bar{z}_{\dot{\alpha}})$. The two-dimensional complexified Heisenberg algebra $[u, v]_\star = 1$ has the Klein operator $k = \cos_\star(\pi v \star u)$, which anti-commutes with u and v and squares to 1. Hence k remains invariant under the canonical $SL(2; \mathbb{C})$ -symmetry that becomes manifest in Weyl order, where the symbol of k is thus proportional to the two-dimensional Dirac delta function. It follows that $(\kappa, \bar{\kappa})$ is invariant under $SL(4; \mathbb{C}) \times \overline{SL}(4; \mathbb{C})$, which is broken by dz^2 and $d\bar{z}^2$ down to a global $GL(2; \mathbb{C}) \times \overline{GL}(2; \mathbb{C})$ symmetry of the Vasiliev system, generated by diagonal $SL(2; \mathbb{C}) \times \overline{SL}(2; \mathbb{C})$ transformations and the exchange $(y_\alpha, z_\alpha) \leftrightarrow (iz_\alpha, -iz_\alpha)$. The latter symmetry is hidden in the formulation in terms of differentials on Z -space while it becomes manifest in the deformed-oscillator formulation.

The integrability is manifest in as much as the associativity of the \star -product in manifest.

The integrability implies the Cartan gauge transformations ⁸

$$\delta_\epsilon A = D\epsilon, \quad \delta_\epsilon \Phi = -[\epsilon, F]_\pi, \quad (\text{D.22})$$

for zero-form gauge parameters $\epsilon(X, P, Z; Y)$ obeying the same kinematic constraints as the master one-form, *i.e.* $\tau(\epsilon) = -\epsilon$ and $(\epsilon)^\dagger = -\epsilon$. The closure of the gauge transformations reads

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{\epsilon_{12}}, \quad \epsilon_{12} = [\epsilon_1, \epsilon_2]_\star, \quad (\text{D.23})$$

defining the algebra $\mathfrak{hs}(4)$.

Finally, the decomposition of the master field equations into unfold and twistor directions read

$$dF + W \star F - F \star \pi(W) = 0, \quad dW + W \star W = 0, \quad (\text{D.24})$$

$$qW + dV + W \star V + V \star W = 0, \quad (\text{D.25})$$

$$qF + V \star F - F \star \pi(V) = 0, \quad qV + V \star V + F \star J = 0. \quad (\text{D.26})$$

The duality unextended system on shell with fixed interaction ambiguities admits the following synthetic form⁹:

$$dA + A \star A + F \star J = 0, \quad dF + A \star F - F \star \pi(A) = 0, \quad (\text{D.27})$$

where (A, Φ) are locally defined differential forms on a noncommutative correspondence space as follows:

$$d = d + q, \quad d = dX^M \partial_M + dP_M \partial^M, \quad q = dZ^\alpha \partial_\alpha, \quad (\text{D.28})$$

$$A = dX^M A_M(\Xi) + dP_M A^M(\Xi) + dZ^\alpha A_\alpha(\Xi), \quad (\text{D.29})$$

$$F = F(\Xi), \quad J = -\frac{i}{4} (b dz^2 \kappa + \bar{b} d\bar{z}^2 \bar{\kappa}), \quad (\text{D.30})$$

where $\Xi^{\overline{M}} = (X^M, P_M, Y^\alpha, Z^\alpha)$ are local coordinates; (X^M, P_M) coordinatize a universal noncommutative phase space; $(Y^\alpha, Z^\alpha) = (y^\alpha, \bar{y}^{\dot{\alpha}}; z^\alpha, -\bar{z}^{\dot{\alpha}})$ coordinatize two mutually commuting copies of the noncommutative twistor space; b and \bar{b} are complex moduli and

$$\kappa := \cos_\star(\pi n), \quad \bar{\kappa} := \cos_\star(\pi \bar{n}), \quad (\text{D.31})$$

⁸These transformations are the canonical transformations of the \star -product algebra generated by D.2 containing the diffeomorphisms of Lagrangian submanifolds of the unfold.

⁹The format applies also to Yang-Mills extended or supersymmetric models; for example, see [51, 52, 53].

with (n, \bar{n}) given by the total chiral number operators in twistor space, are idempotent chiral Klein operators such that

$$dJ = 0, \quad (F, A) \star J - J \star \pi(A, F) = 0, \quad (\text{D.32})$$

where π is the automorphism of the \star -product algebra defined by

$$\pi : (X^M, P_M; y^\alpha, \bar{y}^{\dot{\alpha}}; z^\alpha, \bar{z}^{\dot{\alpha}}) \mapsto (X^M, P_M; -y^\alpha, \bar{y}^{\dot{\alpha}}; -z^\alpha, \bar{z}^{\dot{\alpha}}), \quad d\pi = \pi d. \quad (\text{D.33})$$

The symbols of the Kleinians are distributions on the doubled twistor space whose precise form depend on the choice of ordering scheme (that can thus be adapted to different physical problems); for example, in overall Weyl order they localize to Dirac delta functions (that are useful in trace calculations) while in overall normal order they become Gaussians (that are useful in perturbation theory).

The singular nature of the Kleinians implies that the source term $\Phi \star J$ cannot be absorbed into a field redefinition [33]. Moreover, upon projection of the full equations to a Lagrangian sub-manifold of the universal phase space, say $P_M = 0$, which can be obtained in an expansion in the zero-form, the twistor-space source term induces nontrivial albeit perturbatively defined deformations of the generalized curvatures $dA + A \star A$ and $D\Phi = d\Phi + A \star \Phi - \Phi \star \pi(A)$ of the $hs(4)$ -valued connection $A = A|_{Z=P=0}$ and the twisted-adjoint zero-form $\Phi = \Phi|_{Z=P=0}$. Upon further weak-field expansion around large spin-two gauge fields, *i.e.* vierbein $e^{\alpha\dot{\alpha}}$ and Lorentz connection $(\omega^{\alpha\beta}, \bar{\omega}^{\dot{\alpha}\dot{\beta}})$, the deformations contain the canonical linearized source terms for unfolded Fronsdal tensors in accordance with Vasiliev's central on-shell theorem.

In other words, the Vasiliev system contains a set of nontrivial equations of motion for perturbatively defined Fronsdal tensors. The full system contains, however, various other moduli that have either problematic or no description in terms of Fronsdal fields, such as classical solutions with degenerate vierbeins and topological degrees of freedom contained in the internal connection $A_{\underline{\alpha}}$ [39].

Over and above their formal Cartan integrability, the Vasiliev equations exhibit the following more powerful integrable structures:

- The Maurer-Cartan integrability facilitates the explicit construction of solutions using gauge functions [54, 39, 55, 56, 36] and the formal construction gauge-invariant observables [42];
- The zero-forms $S_\alpha := z_\alpha - 2iA_\alpha$ and $S_{\dot{\alpha}} := \bar{z}_{\dot{\alpha}} - 2iA_{\dot{\alpha}}$ the following generalization of Wigner's

deformed oscillator algebra with local anyonic deformation parameter Φ , viz.

$$\begin{aligned} [S_\alpha, S_\beta]_\star &= -2i\epsilon_{\alpha\beta}(1 - \Phi \star \kappa) \quad , \quad [S_{\dot{\alpha}}, S_{\dot{\beta}}]_\star = -2i\epsilon_{\dot{\alpha}\dot{\beta}}(1 - \Phi \star \bar{\kappa}) \quad , \\ [S_\alpha, S_{\dot{\beta}}]_\star &= 0 \quad , \quad S_\alpha \star \Phi + \Phi \star \pi(S_\alpha) = 0 \quad , \quad S_{\dot{\alpha}} \star \Phi + \Phi \star \bar{\pi}(S_{\dot{\alpha}}) = 0 \quad , \end{aligned} \quad (\text{D.34})$$

which one may also think of as describing the deformation of the symplectic structure on a submanifold of complex dimension two of the doubled twistor space (of complex dimension four).

These properties have been used in a number of circumstances ranging from classical solutions [57, 58, 39, 38, 36] to perturbative calculations of the reduced twistor-space vertices $P(W; \Phi)$ and $J(W, W; \Phi)$ in [52, 52] and direct verification of the Klebanov-Polyakov conjecture [47], first in [59] at the level of cubic scalar self-couplings, and then for the complete cubics in [48, 49].

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An action principle for Vasiliev's four-dimensional higher-spin gravity

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ABSTRACT. We provide Vasiliev's fully nonlinear equations of motion for bosonic higher spin gauge fields in four spacetime dimensions with an action principle. We first extend Vasiliev's original system with differential forms in degrees higher than one. We then derive the resulting duality-extended equations of motion from a variational principle based on a generalized Hamiltonian sigma-model action. The generalized Hamiltonian contains two types of interaction freedoms: One set of functions that appears in the Q-structure of the generalized curvatures of the odd forms in the duality-extended system; and another set depending on the Lagrange multipliers, encoding a generalized Poisson structure, *i.e.* a set of polyvector fields of ranks two or higher in target space. We find that at least one of the two sets of interaction-freedom functions must be linear in order to ensure gauge invariance. We discuss consistent truncations to the minimal Type A and B models (with only even spins), spectral flows on-shell and provide boundary conditions on fields and gauge parameters that are compatible with the variational principle and that make the duality-extended system equivalent, on shell, to Vasiliev's original system.

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1 Introduction

The natural setting for gauge theories with local space-time symmetries is *unfolded dynamics* [1, 2, 3, 4, 5]. The application of this formalism, which is based on *exterior differential systems* (see e.g. [6, 7] and refs. therein), to field theories with local propagating degrees of freedom, such as gravities, supergravities and higher-spin gravities, yields infinite towers of zero-forms that are independent dynamical fields off shell. On shell, their integration constants, or expectation values, represent all the local information of the on-shell curvatures, usually referred to as the Weyl tensors.

In mathematics, an exterior differential system is usually considered as an ideal I in the graded ring of locally defined differential forms on a smooth manifold M that is closed under the operation of exterior differentiation. An integral manifold of a differential system is an immersed submanifold of M on which each form in I restricts to zero. In unfolded dynamics, the generators of I are identified as *generalized curvatures* and the integral manifold becomes a classical solution. Due to Cartan integrability, the curvatures can be integrated and expressed in terms of potentials, providing the fundamental variables in the off-shell formulation.

The canonical framework for the off-shell formulation of unfolded dynamics is based on generalized Poisson sigma models [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19], and [20, 21, 22, 23, 24, 25]. Adapting these models to quasi-topological unfolded systems with infinite towers of zero-forms, provides a framework for quantum field theory that one may refer to as unfolded quantum field theory, or deformation quantum field theory. The resulting key physical question is whether this novel framework actually contains standard relativistic quantum fields; see also [26, 27, 28] for recent developments.¹

Considering retrospectively the works [30, 31, 32, 33, 34], one sees that these formulations of supergravities are examples of unfolded systems, *i.e.* exterior differential systems with infinite towers of Weyl zero-forms, though the locality of supergravity implies that all the dynamic content can be accessed (in the metric phase) by only considering the constraints on the forms in strictly positive degrees, thereby explaining why the authors of [33, 32, 31] did not consider the constraints on the generalized one-form curvatures for the Weyl tensors.

In this paper we shall address this issue by using the fully non-linear and background-independent Vasiliev equations in four spacetime dimensions [2, 35, 36]. These equations possess an algebraic structure that enables us to construct a generalized Hamiltonian action with nontrivial QP -structures, and have geometric structures which allows to construct additional boundary deformations. In this paper we focus on the bulk part of the Hamiltonian action, leaving various deformations on submanifolds to future

¹Note that a relation between the AKSZ formalism and unfolding was not explicitly spelled out before [20]. The observation in [20] mainly relies on the results of [29] where the relation between unfolded and BRST approaches was first established (for linear systems).

works. In fact, already in [3], such an action principle was proposed, which however did not contain any P -structure.

We wish to stress that, unlike the original Fronsdal programme, which attempts to formulate higher-spin gauge theory off shell in a perturbative expansion around constantly curved spacetime, the work in this paper provides a background-independent formulation in terms of master fields living in the correspondence space, *i.e.* the local product of a non-commutative phase-spacetime containing the commutative spacetime as a Lagrangian submanifold and a non-commutative twistor space. Strictly speaking, the Vasiliev system has a huge classical solution space that admits many different perturbative expansions of which only some reduce to Fronsdal systems (with cosmological constant).

2 Duality extension on shell

2.1 Duality extended bosonic models

Our starting point is Vasiliev's on-shell formulation of higher-spin gravity in four spacetime dimensions [2, 35, 36] based on combining free differential algebra and the twistor map (see Appendix D).

Vasiliev's equations of motion provide a particular example of formulation of a classical field theory using free differential algebras, sometimes referred to as unfolded dynamics. In general, unfolded systems can be extended by adding forms in higher degrees. In particular, if the underlying differential algebra contains central and closed elements in degrees $\{0, 2, 4, \dots\}$, also the structure constants can be extended from the real numbers (in degree zero) to general central elements. If this extension is nontrivial, that is, if it cannot be removed by a field redefinition, then we refer to the resulting extended system as a duality extension of the original system. The duality-extended system contains the original system as a consistent subsystem, and this subsystem sources the duality-extended sector via nontrivial couplings involving central elements of positive degrees (see Appendix B for a more detailed discussion).

Vasiliev's equations can be extended adding forms in higher degrees as follows:

$$A = \sum_{p=1,3,\dots} A_{[p]}, \quad B = \sum_{p=0,2,\dots} B_{[p]}, \quad (2.1)$$

where $A_{[p]}$ and $B_{[p]}$ are locally-defined differential forms of total degree p belonging to the algebra of bosonic forms with generic elements

$$f = \sum_{p=0}^{\infty} f_{[p]}(X^M, dX^M; Z^\alpha, dZ^\alpha; Y^\alpha; k, \bar{k}), \quad (2.2)$$

$$f_{[p]}(\lambda dX^M; \lambda dZ^\alpha) = \lambda^p f_{[p]}(dX^M; dZ^\alpha), \quad (2.3)$$

for complex parameters λ (we suppress the irrelevant variables whenever ambiguities cannot arise), where X^M are commuting coordinates, $(Y^\alpha, Z^\alpha) = (y^\alpha, \bar{y}^{\dot{\alpha}}; z^\alpha, \bar{z}^{\dot{\alpha}})$ are non-commutative twistor-space

coordinates and k and \bar{k} are outer Kleinians obeying

$$k \star f = \pi(f) \star k, \quad \bar{k} \star f = \bar{\pi}(f) \star \bar{k}, \quad k \star k = 1 = \bar{k} \star \bar{k}, \quad (2.4)$$

with automorphisms π and $\bar{\pi}$ defined by $\pi d = d\pi$, $\bar{\pi} d = d\bar{\pi}$ and

$$\begin{aligned} \pi[f(z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}})] &= f(-z^\alpha, \bar{z}^{\dot{\alpha}}; -y^\alpha, \bar{y}^{\dot{\alpha}}), \\ \bar{\pi}[f(z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}})] &= f(z^\alpha, -\bar{z}^{\dot{\alpha}}; y^\alpha, -\bar{y}^{\dot{\alpha}}). \end{aligned} \quad (2.5)$$

The bosonic projection and irreducibility conditions amount to

$$\pi\bar{\pi}(f) = f, \quad f = P_+ \star f, \quad \text{where } P_\pm = \frac{1}{2}(1 \pm k \star \bar{k}), \quad (2.6)$$

which implies

$$f = \left[f^{(+)}(X, dX; Z, dZ; Y) + f^{(-)}(X, dX; Z, dZ; Y) \star \frac{(k + \bar{k})}{2} \right] \star P_+. \quad (2.7)$$

The bosonic projection removes all component fields associated with the unfolding of spinorial degrees of freedom in spacetime. Irreducible *minimal* bosonic models can be obtained by imposing reality conditions and discrete symmetries that remove all odd spins; the hermitian conjugation \dagger and the relevant anti-automorphism τ are defined by $d[(\cdot)^\dagger] = [d(\cdot)]^\dagger$, $d\tau = \tau d$,

$$[f(z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}}; k, \bar{k})]^\dagger = \bar{f}(\bar{z}^{\dot{\alpha}}, z^\alpha; \bar{y}^{\dot{\alpha}}, y^\alpha; \bar{k}, k), \quad (2.8)$$

$$\tau[f(z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}}; k, \bar{k})] = f(-iz^\alpha, -i\bar{z}^{\dot{\alpha}}; iy^\alpha, i\bar{y}^{\dot{\alpha}}; k, \bar{k}), \quad (2.9)$$

$$[f_{[p]} \star f'_{[p']}]^\dagger = (-1)^{pp'} (f'_{[p']})^\dagger \star (f_{[p]})^\dagger, \quad \tau[f_{[p]} \star f'_{[p']}] = (-1)^{pp'} \tau(f'_{[p']}) \star \tau(f_{[p]}). \quad (2.10)$$

We shall discuss the minimal models below.

The duality extension of the Vasiliev system is based on the following generalized curvature constraints

$$F + \mathcal{F} = 0, \quad DB = 0, \quad (2.11)$$

with Yang–Mills-like curvature and covariant derivative defined by

$$F = dA + A \star A, \quad DB = dB + A \star B - B \star A, \quad (2.12)$$

and interaction freedom ($I, \bar{I} = 1, 2$)

$$\mathcal{F} = \mathcal{F}_I(B) \star J_{[2]}^I + \mathcal{F}_{\bar{I}}(B) \star J_{[2]}^{\bar{I}} + \mathcal{F}_{I\bar{I}}(B) \star J_{[4]}^{I\bar{I}} \quad (2.13)$$

featuring the central elements

$$(J_{[2]}^I)_{I=1,2} = -\frac{i}{4}(1, k\kappa) \star P_+ \star d^2 z, \quad (J_{[2]}^{\bar{I}})_{\bar{I}=\bar{1},\bar{2}} = -\frac{i}{4}(1, \bar{k}\bar{\kappa}) \star P_+ \star d^2 \bar{z}, \quad (2.14)$$

$$J_{[4]}^{I\bar{I}} = 4i J_{[2]}^I J_{[2]}^{\bar{I}}, \quad (2.15)$$

and \star -functions \mathcal{F}_I , $\mathcal{F}_{\bar{I}}$ and $\mathcal{F}_{I\bar{I}}$ of B such that $\mathcal{F}_I(\lambda)$, $\mathcal{F}_{\bar{I}}(\lambda)$ and $\mathcal{F}_{I\bar{I}}(\lambda)$ ($I, \bar{I} = 1, 2$), viewed as functions of a single complex variable $\lambda \in \mathbb{C}$, are complex analytic in a finite neighborhood of $\lambda = 0$.

The unfolded equations (2.11) are Cartan integrable because the Yang–Mills-like Bianchi identities $DF \equiv 0$ and $DDB \equiv [F, B]_\star$ are compatible with the generalized curvature constraints. In other words, defining the generalized curvatures

$$\mathcal{R}^A = F + \mathcal{F}, \quad \mathcal{R}^B = DB, \quad (2.16)$$

one has the generalized Bianchi identities

$$D\mathcal{R}^A - (\mathcal{R}^B \partial_B) \star \mathcal{F} \equiv 0, \quad D\mathcal{R}^B - [\mathcal{R}^A, B]_\star \equiv 0. \quad (2.17)$$

The potentials $\{A_{[1]}, B_{[2]}, A_{[3]}, B_{[4]}, \dots\}$ in positive form degree share one and the same Weyl zero-form $B_{[0]}$, that hence contain all the local perturbative degrees of freedom of the extended system. One may refer to $\{B_{[0]}, A_{[1]}, B_{[2]}, A_{[3]}, B_{[4]}, \dots\}$ as a duality extension of the original Vasiliev system consisting of $\{B_{[0]}, A_{[1]}\}$ in the sense that the presence of the central elements in degree four implies that $\{B_{[2]}, A_{[3]}, B_{[4]}, \dots\}$ cannot in general be set equal to zero on shell. Moreover, the extension is massless in the sense that for each $p \in \{1, 2, 3, \dots\}$ the system of forms with degrees $p' \leq p$ constitutes a closed subsystem, *i.e.* their curvatures do not depend on the forms with degrees $p' > p$. In particular, this means that any (locally-defined) exact solution to the duality extended system contains a (locally-defined) exact solution to the original Vasiliev system. The converse statement requires a more careful analysis that we defer here.

2.2 A duality extended spectral flow

The duality extended system possesses a spectral flow [37] describing the evolution of the system on shell under changes in a vacuum expectation value ν and a coupling g defined by the field redefinition

$$B = \nu \mathbf{1} + gB'. \quad (2.18)$$

We stress that the parameters (g, ν) are part of the moduli space of the unfolded equations of motion, that is, both A and B depend on (g, ν) on shell and in such a way that the differential d commutes with $(\partial_g, \partial_\nu)$. Letting $f = f(A, dA, B, dB)$ and defining the flow operator

$$L_1 f = \partial_g f - \mu_1 B' \star \partial_\nu f - \partial_\nu f \star \mu_2 B', \quad \mu_1, \mu_2 \in \mathbb{C}, \quad \mu_1 + \mu_2 = 1, \quad (2.19)$$

one has

$$L_1 F \equiv DL_1 A + \mu_1 DB' \star \partial_\nu A - \mu_2 \partial_\nu A \star DB', \quad (2.20)$$

$$L_1 DB \equiv DL_1 B + [L_1 A, B]_\star + \mu_1 DB \star \partial_\nu B' + \mu_2 \partial_\nu B' \star DB, \quad (2.21)$$

$$L_1 \mathcal{F} \equiv (L_1 B \partial_B) \star \mathcal{F}. \quad (2.22)$$

It follows that the duality extended equations of motion are compatible with the flow equations

$$L_1 A \approx 0, \quad L_1 B \approx 0, \quad (2.23)$$

where the last flow equation is equivalent to that $L_1 B' \approx 0$.

The flow equations generalize as follows: one first redefines

$$B = \nu + \mathcal{N}(B'), \quad \mathcal{N} = \nu_1 g B' + \nu_2 g^2 B'^{\star 2} + \nu_3 g^3 B'^{\star 3} + \dots, \quad (2.24)$$

where ν_k ($k \geq 1$) are constants and g the coupling. The flow operator defined by

$$L f = \partial_g f - \mathcal{M}_1(B') \star \partial_\nu f - \partial_\nu f \star \mathcal{M}_2(B'), \quad (2.25)$$

where the two \star -functions defined by ($i = 1, 2$)

$$\mathcal{M}_i = \mu_{i,1} g B' + \mu_{i,2} g^2 B'^{\star 2} + \dots, \quad \mu_{1,k} + \mu_{2,k} = k \nu_k \quad (k \geq 1); \quad (2.26)$$

obey

$$L \mathcal{F} \equiv (L B \partial_B) \star \mathcal{F}, \quad (2.27)$$

$$L B = \nu_1 L B' + \nu_2 g^2 (L B' \star B' + B' \star L B') + \dots, \quad (2.28)$$

$$L F = D L A + D \mathcal{M}_1 \star \partial_\nu A - \partial_\nu A \star D \mathcal{M}_2, \quad (2.29)$$

$$L D B' = D L B' + [L A, B']_\star + D \mathcal{M}_1 \star \partial_\nu B' + \partial_\nu B' \star D \mathcal{M}_2, \quad (2.30)$$

and it follows that one can set the constraints

$$L A = 0, \quad L B' = 0, \quad (2.31)$$

where the latter constraint thus implies that $L B = 0$. One can redefine $\mathcal{N} = g B'$ so that $\nu_1 = 1$ and $\nu_k = 0$ for $k > 1$, leaving the freedom in \mathcal{M}_i that generalizes the two-parameter freedom in having μ_1 and μ_2 .

2.3 Consistent truncations

There are two possible reality conditions leading to models with negative cosmological constant $\Lambda < 0$, that we parameterize using $\epsilon_{\mathbb{R}} = \pm 1$ as follows:

$$(A_{[p]})^\dagger = -(\epsilon_{\mathbb{R}})^{\frac{p-1}{2}} A_{[p]}, \quad (B_{[p]})^\dagger = (\epsilon_{\mathbb{R}})^{\frac{p}{2}} B_{[p]}, \quad (2.32)$$

$$(\mathcal{F}_I(\lambda))^\dagger = \mathcal{F}_{\bar{I}}(\lambda^\dagger), \quad \mathcal{F}_{I\bar{J}}(\lambda)^\dagger = \epsilon_{\mathbb{R}} \mathcal{F}_{J\bar{I}}(\lambda^\dagger). \quad (2.33)$$

Moreover, using the map

$$\pi_k : (k, \bar{k}) \mapsto (-k, -\bar{k}) , \quad (2.34)$$

there are two possible projections to models without topological (adjoint) zero-forms, that we parameterize using $\epsilon_k = \pm 1$ as follows:

$$\pi_k(A_{[p]}) = (\epsilon_k)^{\frac{p-1}{2}} A_{[p]} , \quad \pi_k(B_{[p]}) = -(\epsilon_k)^{\frac{p}{2}} B_{[p]} , \quad (2.35)$$

$$\mathcal{F}_I(-\lambda) = (-1)^{I+1} \mathcal{F}_I(\lambda) , \quad \mathcal{F}_{I\bar{I}}(-\lambda) = (-1)^{I+\bar{I}} \epsilon_k \mathcal{F}_{I\bar{I}}(\lambda) . \quad (2.36)$$

Using the parity transformation P defined by $Pd = dP$ and

$$P[f(X^M; z^\alpha, \bar{z}^{\dot{\alpha}}; y^\alpha, \bar{y}^{\dot{\alpha}}; k, \bar{k})] = (Pf)(X^M; -\bar{z}^{\dot{\alpha}}, -z^\alpha; \bar{y}^{\dot{\alpha}}, y^\alpha; \bar{k}, k) , \quad (2.37)$$

which is an automorphism of the \star -product algebra and where Pf is expanded in terms of parity reversed component fields, there are four ways of fixing parities, that we parameterize using $\epsilon, \tilde{\epsilon} = \pm 1$ as follows:

$$P(A_{[p]}) = (\epsilon \tilde{\epsilon})^{\frac{p-1}{2}} A_{[p]} , \quad P(B_{[p]}) = (\epsilon)^{\frac{p+2}{2}} (\tilde{\epsilon})^{\frac{p}{2}} B_{[p]} , \quad (2.38)$$

$$\mathcal{F}_{\bar{I}}(\lambda) = \mathcal{F}_I(\epsilon \lambda) , \quad \mathcal{F}_{I\bar{J}}(\lambda) = \epsilon \tilde{\epsilon} \mathcal{F}_{J\bar{I}}(\epsilon \lambda) . \quad (2.39)$$

Finally, the τ -projection to the minimal models with only even propagating spins reads

$$\tau(A_{[p]}) = (-1)^{\frac{p+1}{2}} A_{[p]} , \quad \tau(B_{[p]}) = (-1)^{\frac{p}{2}} B_{[p]} , \quad (2.40)$$

which is the unique choice since $\tau(J_{[p]}) = (-1)^{\frac{p}{2}} J_{[p]}$ (and there is no condition on \mathcal{F}).

In the $(B_{[0]}, A_{[1]})$ -sector, which forms a closed subsystem, the assignement of k -parity combined with the freedom in redefining A_α can be used to replace [2]

$$(\mathcal{F}_1, \mathcal{F}_2; \mathcal{F}_{\bar{1}}, \mathcal{F}_{\bar{2}}) \rightarrow (0, (1 - \mathcal{F}_1)^{\star(-1)} \star \mathcal{F}_2; 0, (1 - \mathcal{F}_{\bar{1}})^{\star(-1)} \star \mathcal{F}_{\bar{2}}) . \quad (2.41)$$

Imposing also reality and parity conditions, of which the latter is a multiple choice parametrized by $\epsilon = \pm 1$, the remaining interaction function $(1 - \mathcal{F}_1)^{\star(-1)} \star \mathcal{F}_2$ becomes real and odd, hence defining the new master field

$$\Phi \star P_+ = (1 - \mathcal{F}_1)^{\star(-1)} \star \mathcal{F}_2 \star k \star P_+ , \quad (2.42)$$

obeying the twisted reality condition $(\Phi)^\dagger = \pi(\Phi)$ and the parity condition $P(\Phi) = \epsilon \Phi$ leading to a physical scalar that is even under parity for $\epsilon = 1$ and odd under parity for $\epsilon = -1$. Finally, one may project out the odd spins by imposing $\tau(\Phi) = \pi(\Phi)$ yielding the minimal bosonic models.

Assuming linear interaction functions

$$\mathcal{F}_I = b_I B , \quad \mathcal{F}_{\bar{I}} = b_{\bar{I}} B , \quad \mathcal{F}_{I\bar{I}} = c_{I\bar{I}} B , \quad (2.43)$$

and defining a total central element

$$J = J_{[2]} + J_{[4]} \quad (2.44)$$

via

$$B \star J_{[2]} = \mathcal{F}_I \star J_{[2]}^I + \mathcal{F}_{\bar{I}} \star J_{[2]}^{\bar{I}} \quad , \quad B \star J_{[4]} = \mathcal{F}_{I\bar{I}} \star J_{[4]}^{I\bar{I}} \quad , \quad (2.45)$$

$$J_{[2]} = -\frac{i}{4} \left[dz^2(b_1 + b_2 k \kappa) + d\bar{z}^2(b_{\bar{1}} + b_{\bar{2}} \bar{k} \bar{\kappa}) \right] \star P_+ \quad , \quad (2.46)$$

$$J_{[4]} = -\frac{i}{4} dz^2 d\bar{z}^2 \left[c_{1\bar{1}} + c_{2\bar{1}} k \kappa + c_{1\bar{2}} \bar{k} \bar{\kappa} + c_{2\bar{2}} \kappa \bar{\kappa} \right] \star P_+ \quad , \quad (2.47)$$

the reality, k -parity and P -parity conditions imply

$$(J_{[p]})^\dagger = -(\epsilon_{\mathbb{R}})^{\frac{p-2}{2}} J_{[p]} \quad , \quad \pi_k(J_{[p]}) = -(\epsilon_k)^{\frac{p-2}{2}} J_{[p]} \quad , \quad P(J_{[p]}) = (\epsilon)^{\frac{p}{2}} (\tilde{\epsilon})^{\frac{p-2}{2}} J_{[p]} \quad , \quad (2.48)$$

which constrain the parameters $(b_I, b_{\bar{I}}, c_{I\bar{I}})$. These conditions admit nontrivial solutions for $J_{[p]}$ for all combinations of signs except for $\epsilon_k = \tilde{\epsilon} = -1$ since $\epsilon_k = -1$ implies that $\tilde{\epsilon} = +1$.

3 Generalized Hamiltonian action principle

3.1 Graded cyclic chiral trace

Vasiliev's equations are formulated in terms of master fields which one may think of as functions on a total space called *correspondance space* \mathfrak{C} , that is locally a product space $M_\xi \times \mathfrak{Z} \times \mathfrak{Y}$ where \mathfrak{Z} and \mathfrak{Y} are two copies of a non-commutative twistor space and M_ξ denotes a coordinate chart of a commuting base manifold M , see Appendix D for more details. In order to build an action principle, we need to integrate over the correspondance space. The integration over \mathfrak{C} of a globally defined $(\hat{p} + 1)$ -form \mathcal{L} is defined by

$$\int_{\mathfrak{C}} \mathcal{L} = \sum_{\xi} \int_{M_\xi} \text{Tr} [f_{\mathcal{L}}] \quad , \quad (3.1)$$

where $f_{\mathcal{L}}$ denotes a symbol of \mathcal{L} and the chiral trace operation is defined by

$$\text{Tr} [f] = \sum_m \int_{\mathfrak{Z} \times \mathfrak{Y}} \frac{d^2 y d^2 \bar{y}}{(2\pi)^2} \frac{f_{[m;2,2]}|_{k=0=\bar{k}}}{(2\pi)^2} \quad , \quad (3.2)$$

using the decomposition $f_{[p]} = \sum_{\substack{m+q+\bar{q}=p \\ q, \bar{q} \leq 2}} f_{[m;q,\bar{q}]}$ with

$$f_{[m;q,\bar{q}]}(\lambda dX^M; \mu dz^\alpha, \bar{\mu} d\bar{z}^{\dot{\alpha}}) = \lambda^m \mu^q \bar{\mu}^{\bar{q}} f_{[m;q,\bar{q}]}(dX^M; dz^\alpha, d\bar{z}^{\dot{\alpha}}) \quad , \quad (3.3)$$

and with integration domain consisting of real contours for $\{y^\alpha, z^\alpha\}$ and $\{\bar{y}^{\dot{\alpha}}, \bar{z}^{\dot{\alpha}}\}$, respectively, that is, one performs separate integrations over the holomorphic and anti-holomorphic variables treated as

independent real variables (for related discussions, see *e.g.* Appendix G of [38]). The choice of the chiral integration domain (instead of the complex integration domain) implies that

$$\mathrm{Tr} [\pi(f)] = \mathrm{Tr} [\bar{\pi}(f)] = \mathrm{Tr} [f] , \quad (3.4)$$

which in its turn implies graded cyclicity,

$$\mathrm{Tr} [f_{[p]} \star f'_{[p']}] = (-1)^{pp'} \mathrm{Tr} [f'_{[p']} \star f_{[p]}] , \quad (3.5)$$

as can be seen by expanding $f_{[p]} = (f_{[p]}^{(+)} + f_{[p]}^{(-)} \star k) \star P_+ \text{ idem } f'_{[p']}$ which yields

$$\mathrm{Tr} [f_{[p]} \star f'_{[p']}] = \frac{1}{2} \mathrm{Tr} [f_{[p]}^{(+)} \star f'_{[p']}^{(+)} + f_{[p]}^{(-)} \star \pi(f'_{[p']}^{(-)})] , \quad (3.6)$$

where the second term is graded cyclic by virtue of the chiral integration. Furthermore, the chiral trace operation commutes to hermitian conjugation and is invariant under P and π_k ,

$$(\mathrm{Tr} [f])^\dagger = \mathrm{Tr} [(f)^\dagger] , \quad \mathrm{Tr} [P(f)] = \mathrm{Tr} [f] , \quad \mathrm{Tr} [\pi_k(f)] = \mathrm{Tr} [f] . \quad (3.7)$$

Finally, one may seek to impose boundary conditions in $\mathfrak{Z} \times \mathfrak{Y}$ such that the integration contours can be rotated from real to imaginary axes in the sense that

$$\mathrm{Tr} [\tau(f)] = \mathrm{Tr} [f] . \quad (3.8)$$

We shall finally assume that the integration over \mathfrak{C} is non-degenerate such that if $\mathrm{Tr} [f \star g] = 0$ for all f then $g = 0$. It is an interesting open problem to understand whether the π , P and τ symmetries could be violated on classical observables evaluated on exact solutions that one may seek to interpret as describing topology changes of the twistor space which we leave for future studies [39]. In what follows, we shall always assume that the discrete symmetries hold off shell.

3.2 Odd-dimensional bulk ($\hat{p} \in 2\mathbb{N}$)

3.2.1 Action principle

In the case of an odd-dimensional base manifold of dimension $\hat{p}+1 = 2n+5$ with $n \in \{0, 1, 2, \dots\}$ such that $\dim(M) = 2n+1$, the duality-extended equations of motion follow from the variational principle based on the generalized Hamiltonian bulk action

$$S_{\mathrm{bulk}}^{\mathrm{cl}}[\{A, B, U, V\}_\xi] = \sum_\xi \int_{M_\xi} \mathrm{Tr} \left[U \star DB + V \star \left(F + \mathcal{G}(B, U; J^I, J^{\bar{I}}, J^{I\bar{I}}) \right) \right] , \quad (3.9)$$

with interaction freedom \mathcal{G} and locally-defined master fields decomposing under total form degree into

$$A = A_{[1]} + A_{[3]} + \dots + A_{[2m-1]} , \quad B = B_{[0]} + B_{[2]} + \dots + B_{[2m-2]} , \quad (3.10)$$

$$U = U_{[2]} + U_{[4]} + \cdots + U_{[2m]}, \quad V = V_{[1]} + V_{[3]} + \cdots + V_{[2m-1]}, \quad m = n + 2. \quad (3.11)$$

The function \mathcal{G} must be constrained in order for the action to be gauge invariant and in order to avoid systems that are trivial. In what follows, we shall consider the special case

$$\mathcal{G} = \mathcal{F}(B; J^I, J^{\bar{I}}, J^{I\bar{I}}) + \widetilde{\mathcal{F}}(U; J^I, J^{\bar{I}}, J^{I\bar{I}}), \quad (3.12)$$

$$\mathcal{F} = \mathcal{F}_I(B) \star J_{[2]}^I + \mathcal{F}_{\bar{I}}(B) \star J_{[2]}^{\bar{I}} + \mathcal{F}_{I\bar{I}}(B) \star J_{[4]}^{I\bar{I}}, \quad (3.13)$$

$$\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_0(U) + \widetilde{\mathcal{F}}_I(U) \star J_{[2]}^I + \widetilde{\mathcal{F}}_{\bar{I}}(U) \star J_{[2]}^{\bar{I}} + \widetilde{\mathcal{F}}_{I\bar{I}}(U) \star J_{[4]}^{I\bar{I}}, \quad (3.14)$$

where the (non-)vanishing of the coupling $\lambda := \partial_U \widetilde{\mathcal{F}}_0|_{U=0}$ implies that the target space is equipped with a Poisson (symplectic) structure. In the case of a proper Poisson structure with $\lambda = 0$ the action cannot be written as a boundary term.

Denoting $Z^i = (A, B, U, V)$, the general variation of the action defines generalized curvatures \mathcal{R}^i as follows:

$$\delta S = \sum_{\xi} \int_{M_{\xi}} \text{Tr} [\mathcal{R}^i \star \delta Z^j \mathcal{O}_{ij}] + \sum_{\xi} \int_{\partial M_{\xi}} \text{Tr} [U \star \delta B - V \star \delta A], \quad (3.15)$$

where one thus has

$$\mathcal{R}^A = F + \mathcal{F} + \widetilde{\mathcal{F}}, \quad \mathcal{R}^B = DB + (V \partial_U) \star \widetilde{\mathcal{F}}, \quad (3.16)$$

$$\mathcal{R}^U = DU - (V \partial_B) \star \mathcal{F}, \quad \mathcal{R}^V = DV + [B, U]_{\star}, \quad (3.17)$$

with \mathcal{O}_{ij} being a constant non-degenerate matrix (defining a symplectic form of degree $\hat{p} + 2$ on the \mathbb{N} -graded target space of the bulk theory). Treating Z^i and dZ^i as independent variables, one has the differential identities

$$D\mathcal{R}^A - (\mathcal{R}^B \partial_B) \star \mathcal{F} - (\mathcal{R}^U \partial_U) \star \widetilde{\mathcal{F}} \equiv \mathcal{A}^A, \quad (3.18)$$

$$D\mathcal{R}^B - [\mathcal{R}^A, B]_{\star} - (\mathcal{R}^V \partial_U) \star \widetilde{\mathcal{F}} - (\mathcal{R}^U \partial_U) \star (V \partial_U) \star \widetilde{\mathcal{F}} \equiv \mathcal{A}^B, \quad (3.19)$$

$$D\mathcal{R}^U - [\mathcal{R}^A, U]_{\star} + (\mathcal{R}^V \partial_B) \star \mathcal{F} + (\mathcal{R}^B \partial_B) \star (V \partial_B) \star \widetilde{\mathcal{F}} \equiv \mathcal{A}^U, \quad (3.20)$$

$$D\mathcal{R}^V - [\mathcal{R}^A, V]_{\star} - [\mathcal{R}^B, U]_{\star} + [\mathcal{R}^U, B]_{\star} \equiv \mathcal{A}^V, \quad (3.21)$$

with dZ^i -independent quantities $\mathcal{A}^i \equiv \mathcal{A}^i(Z^j)$ given by

$$\mathcal{A}^A \equiv -((V \partial_U) \star \widetilde{\mathcal{F}}) \partial_B \star \mathcal{F} + ((V \partial_B) \star \mathcal{F}) \partial_U \star \widetilde{\mathcal{F}}, \quad (3.22)$$

$$\mathcal{A}^B \equiv ((V \partial_B) \star \mathcal{F}) \partial_U \star (V \partial_U) \star \widetilde{\mathcal{F}}, \quad (3.23)$$

$$\mathcal{A}^U \equiv ((V \partial_U) \star \widetilde{\mathcal{F}}) \partial_B \star (V \partial_B) \star \mathcal{F}, \quad (3.24)$$

$$\mathcal{A}^V \equiv 0, \quad (3.25)$$

where the last identity follows from

$$[U, (V\partial_U) \star \widetilde{\mathcal{F}}]_\star \equiv -[V, \widetilde{\mathcal{F}}]_\star, \quad [B, (V\partial_B) \star \mathcal{F}]_\star \equiv -[V, \mathcal{F}]_\star. \quad (3.26)$$

The quantities \mathcal{A}^i thus represent obstructions to generalized Bianchi identities off shell and hence to Cartan integrability of the unfolded equations of motion $\mathcal{R}^i \approx 0$, where in this Section we use weak equalities for equations that hold on shell. These obstructions vanish identically (without further algebraic constraints on Z^i) in at least the following two cases:

$$\text{bilinear } Q\text{-structure} : \mathcal{F} = B \star J, \quad J = J_{[2]} + J_{[4]}, \quad (3.27)$$

$$\text{bilinear } P\text{-structure} : \widetilde{\mathcal{F}} = U \star J', \quad J' = J'_{[2]} + J'_{[4]}, \quad (3.28)$$

where the central elements are expanded as in Eqs. (2.44)–(2.47).

At this stage it is useful to recall (see Appendix C) that if $\mathcal{R}^i = dZ^i + \mathcal{Q}^i(Z^j)$ defines a set of generalized curvatures, then one has the following three equivalent statements: (i) \mathcal{R}^i obey a set of generalized Bianchi identities $d\mathcal{R}^i - (\mathcal{R}^j \partial_j) \star \mathcal{Q}^i \equiv 0$; (ii) \mathcal{R}^i transform into each other under Cartan gauge transformations $\delta_\varepsilon Z^i = d\varepsilon^i - (\varepsilon^j \partial_j) \star \mathcal{Q}^i$; and (iii) the quantity $\vec{\mathcal{Q}} := \mathcal{Q}^i \partial_i$ is a Q -structure, *i.e.* a nilpotent \star -vector field of degree one in target space, *viz.* $\vec{\mathcal{Q}} \star \mathcal{Q}^i \equiv 0$. Furthermore, in the case of differential algebras on commutative base manifolds, one can show that if \mathcal{R}^i are defined via a variational principle as in (3.15) (with constant \mathcal{O}_{ij}), then the action S remains invariant under $\delta_\varepsilon Z^i$.

In the two Cartan-integrable cases at hand, one thus has the on-shell Cartan gauge transformations

$$\delta_{\epsilon, \eta} A = D\epsilon^A - (\epsilon^B \partial_B) \star \mathcal{F} - (\eta^U \partial_U) \star \widetilde{\mathcal{F}}, \quad (3.29)$$

$$\delta_{\epsilon, \eta} B = D\epsilon^B - [\epsilon^A, B]_\star - (\eta^V \partial_U) \star \widetilde{\mathcal{F}} - (\eta^U \partial_U) \star (V\partial_U) \star \widetilde{\mathcal{F}}, \quad (3.30)$$

$$\delta_{\epsilon, \eta} U = D\eta^U - [\epsilon^A, U]_\star + (\eta^V \partial_B) \star \mathcal{F} + (\epsilon^B \partial_B) \star (V\partial_B) \star \mathcal{F}, \quad (3.31)$$

$$\delta_{\epsilon, \eta} V = D\eta^V - [\epsilon^A, V]_\star - [\epsilon^B, U]_\star + [\eta^U, B]_\star. \quad (3.32)$$

These transformations remain symmetries off shell as can be seen using the following set of identities:

$$\text{bilinear } P\text{-structure} : \text{Tr} [J' \star V \star (V\partial_B) \star (\epsilon^B \partial_B) \star \mathcal{F}] \equiv 0, \quad (3.33)$$

$$\text{Tr} [V \star (DB\partial_B) \star (\epsilon^B \partial_B) \star \mathcal{F} + DB \star (V\partial_B) \star (\epsilon^B \partial_B) \star \mathcal{F}] \equiv 0,$$

$$\text{Tr} [\eta^V \star (DB\partial_B) \star \mathcal{F} - DB \star (\eta^V \partial_B) \star \mathcal{F}] \equiv 0, \quad (3.34)$$

$$\text{bilinear } Q\text{-structure} : \text{Tr} [J \star V \star (V\partial_U) \star (\eta^U \partial_U) \star \widetilde{\mathcal{F}}] \equiv 0, \quad (3.35)$$

$$\text{Tr} [V \star (DU\partial_U) \star (\eta^U \partial_U) \star \widetilde{\mathcal{F}} + DU \star (V\partial_U) \star (\eta^U \partial_U) \star \widetilde{\mathcal{F}}] \equiv 0,$$

$$\text{Tr} [\eta^V \star (DU\partial_U) \star \widetilde{\mathcal{F}} - DU \star (\eta^V \partial_U) \star \widetilde{\mathcal{F}}] \equiv 0. \quad (3.36)$$

More precisely, the (ϵ^A, ϵ^B) -symmetries leave the Lagrangian invariant while the (η^U, η^V) -symmetries transform the Lagrangian into a nontrivial total derivative, *viz.*

$$\delta_{\epsilon, \eta} \mathcal{L} \equiv d \left(\text{Tr} \left[\eta^U \star \mathcal{K}_U + \eta^V \star \mathcal{K}_V \right] \right), \quad (3.37)$$

for $(\mathcal{K}_U, \mathcal{K}_V)$ that are not identically zero. It follows that the Cartan gauge algebra \mathfrak{g} is of the form

$$\mathfrak{g} \cong \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with $\mathfrak{g}_1 \cong \text{span}\{\epsilon^A, \epsilon^B\}$ and $\mathfrak{g}_2 \cong \text{span}\{\eta^U, \eta^V\}$, as one can verify explicitly using the formulae (C.15) given in Appendix C.

3.2.2 Global formulation, boundary conditions and embedding of Vasiliev's original system

Exponentiation of the infinitesimal Cartan gauge transformations leads to locally-defined gauge orbits consisting of elements (see Appendix A)

$$Z_{\lambda, d\lambda; Z_0}^i = \mathcal{G}_{\lambda, d\lambda; Z} \star Z^i|_{Z^i=Z_0^i}, \quad (3.38)$$

$$\mathcal{G}_{\lambda, d\lambda; Z} := \exp_\star \overrightarrow{\mathcal{F}}_{\lambda, d\lambda; Z}, \quad \overrightarrow{\mathcal{F}}_{\lambda, d\lambda; Z} := (d\lambda^i - (\lambda^j \partial_j) \star \mathcal{Q}^i) \frac{\partial}{\partial Z^i}, \quad (3.39)$$

where λ^i and Z_0^i , respectively, are gauge functions and representatives of the orbits defined in coordinate charts of the base manifold. On shell, one has

$$dZ_0^i + \mathcal{Q}^i(Z_0^j) \approx 0 \quad \Rightarrow \quad dZ_{\lambda, d\lambda; Z_0}^i + \mathcal{Q}^i(Z_{\lambda, d\lambda; Z_0}^j) \approx 0, \quad (3.40)$$

as can be seen by first writing $d \approx \overrightarrow{\mathcal{F}}_{d\lambda} - \overrightarrow{\mathcal{Q}}$ where $\overrightarrow{\mathcal{F}}_{d\lambda} := d\lambda^i \partial / \partial \lambda^i$ and $\overrightarrow{\mathcal{Q}} := \mathcal{Q}^i \partial / \partial Z^i$, and then using $[\overrightarrow{\mathcal{F}}_{d\lambda} - \overrightarrow{\mathcal{Q}}, \overrightarrow{\mathcal{F}}_{\lambda, d\lambda; Z}]_\star \equiv 0$ and $[\exp_\star \overrightarrow{\mathcal{X}}]_\star (\mathcal{F} \star \mathcal{F}') \equiv ([\exp_\star \overrightarrow{\mathcal{X}}] \star \mathcal{F}) \star ([\exp_\star \overrightarrow{\mathcal{X}}] \star \mathcal{F}')$ for any \star -vector field $\overrightarrow{\mathcal{X}}$ and \star -functions \mathcal{F} and \mathcal{F}' (see Appendix C for details).

In particular, it follows that the space of (locally-defined) classical solutions to the duality extended $(A, B; U, V)$ -system contains a subspace of (locally-defined) classical solutions to the duality extended (A, B) -system, obtained simply by setting $U = 0 = V$. The (A, B) -system contains in its turn a subset of the (locally-defined) solutions to the original Vasiliev system in form degrees 0 and 1. The converse issue, whether any given (locally-defined) exact solution to the original Vasiliev system can be uplifted to the (A, B) -system, requires, however, a more careful analysis of the gauge orbits in degrees greater than 1 (due to the non-polynomial dependencies on the integration constants for the Weyl zero-form and the zero-form gauge functions).

Turning to the global formulation, it follows from Eq. (3.37) that the gauge parameters $(\epsilon_\xi^A, \epsilon_\xi^B) \in \mathfrak{g}_1$ can be locally defined on M , that is, defined independently on the coordinate charts M_ξ — provided that the action is not perturbed by impurities that break some of the (ϵ^A, ϵ^B) -symmetries, as for example

in the soldered phase where perturbations break the local translations in $\epsilon^{A[1]}$. From Eq. (3.37) it also follows that $(\eta^U, \eta^V) \in \mathfrak{g}_2$ need to be defined globally on M , that is, $(\eta^U, \eta^V)|_\xi$ and $(\eta^U, \eta^V)|_{\xi'}$ must be related by transition functions $\{t_\xi^{\xi'}\}$ across the chart boundary between M_ξ and $M_{\xi'}$; in practice this means that one may take (η_ξ^U, η_ξ^V) to have compact support in M_ξ .

The unbroken phase of the theory thus consists of local representatives $Z_\xi^i = (A, B; U, V)|_\xi$ defined up to gauge transformations with parameters $(\epsilon_\xi^A; \epsilon_\xi^B)$ that are unrestricted on ∂M_ξ and parameters (η_ξ^U, η_ξ^V) with the aforementioned restrictions on ∂M_ξ , with transitions of the form

$$Z_\xi^i = \mathcal{G}_\xi^{\xi'} \star Z_{\xi'}^i \quad \text{defined on } M_\xi \cap M_{\xi'}, \quad (3.41)$$

where $\mathcal{G}_\xi^{\xi'} = \exp_\star \overrightarrow{\mathcal{F}}_{t, dt; Z}|_\xi^{\xi'}$ with transition functions $t_\xi^{\xi'} \in \mathfrak{g}_1$ defined on $M_\xi \cap M_{\xi'}$.

More generally, softly broken phases of the theory arise by taking the transition functions $\{t_\xi^{\xi'}\}$ to be generated by various unbroken subalgebras $\mathfrak{l} \subseteq \mathfrak{g}_1$. Their moduli spaces $\mathcal{M}_\mathfrak{l}$ can be coordinatized by classical observables $\mathcal{O}_\mathfrak{l}$ that are manifestly \mathfrak{l} -invariant off shell and diffeomorphism-invariant on shell (one may thus think of the unbroken phase $\mathcal{M}_\mathfrak{g}$ as the smallest homotopy phase for a given base manifold; it can be embedded into various broken phases). Of particular interest is the soldered phase in which the action is perturbed as to softly break the gauge symmetries associated with the π -odd projection of $A_{[1]}$. The unbroken gauge algebra in this case thus consists of the π -even projection $\frac{1}{2}(1 + \pi)\epsilon^{A[1]}$ together with the remaining ϵ -parameters of positive form degree.

Hence, to achieve a globally well-defined variational principle, one considers globally-defined field configurations off shell consisting of locally-defined representatives $\{Z_\xi^i\}$ related on chart boundaries via transitions (3.41) for a given structure algebra $\mathfrak{l} \subseteq \mathfrak{g}_1$. The manifest \mathfrak{g}_1 -invariance implies that in the general variation (3.15), the contributions from two adjacent boundaries ∂M_ξ and $\partial M_{\xi'}$ cancel; on such a boundary one has the transition functions $(t \equiv t_\xi^{\xi'})$

$$\delta_t(\delta A) = -[t^A, \delta A]_\star - (\delta B \partial_B) \star (t^B \partial_B) \star \mathcal{F}, \quad (3.42)$$

$$\delta_t(\delta B) = -[t^A, \delta B]_\star + \{t^B, \delta A\}_\star, \quad (3.43)$$

$$\delta_t U = -[t^A, U]_\star + (t^B \partial_B) \star (V \partial_B) \star \mathcal{F}, \quad (3.44)$$

$$\delta_t V = -[t^A, V]_\star - [t^B, U]_\star, \quad (3.45)$$

which implies that $(t \equiv t_\xi^{\xi'})$

$$\delta_t \left(\int_{\partial M_\xi} \text{Tr} [U \star \delta B - V \star \delta A] \right) \quad (3.46)$$

$$= \int_{\partial M_\xi} \text{Tr} [V \star (\delta B \partial_B) \star (t^B \partial_B) \star \mathcal{F} - \delta B \star (V \partial_B) \star (t^B \partial_B) \star \mathcal{F}] \equiv 0. \quad (3.47)$$

One is thus left with contributions from true boundaries $\partial M_\xi \subset \partial M$ (including boundaries of homotopy cylinders surrounding impurities of co-dimension greater than one). It follows that the natural boundary conditions compatible with the locally-defined gauge symmetries are the Dirichlet conditions

$$(U, V)|_{\partial M} = 0. \quad (3.48)$$

In summary, a classical solution can thus be specified by fixing

- (i) the transition functions $\{t_\xi^{\xi'}\} \in \mathfrak{l} \subseteq \mathfrak{g}_1$;
- (ii) an initial datum for the zero-form $B_{[0]}$, say

$$B_{[0]}|_p = C(Y; k, \bar{k}) , \quad (3.49)$$

at some given point $p \in \mathfrak{B}$ in the base manifold;

- (iii) boundary conditions on the gauge functions associated with the softly-broken gauge symmetries, *viz.*

$$\lambda|_{\partial M} = 0 \quad \text{for } \lambda \in \mathfrak{g}_1/\mathfrak{l} ; \quad (3.50)$$

and

- (iv) the boundary conditions (3.48) on the Lagrange multipliers.

3.2.3 Duality extended spectral flow with Lagrange multipliers

The equations of motion $\mathcal{R}^i \approx 0$ of the extended Lagrangian system $Z^i = (A, B; U, V)$ with bilinear P and Q structures (*i.e.* linear \mathcal{F} and $\widetilde{\mathcal{F}}$ functions) are compatible with the extended flow equations $L_1 A \approx 0 \approx L_1 B$ (or equivalently $L_1 B' \approx 0$) and

$$L_1 U \approx \mu_1 V' \star (\partial_\nu A) - \mu_2 (\partial_\nu A) \star V' , \quad L_1 V' \approx \mu_1 V' \star (\partial_\nu B') + \mu_2 (\partial_\nu B') \star V' , \quad (3.51)$$

with flow operator L_1 given by (2.19) and the redefinition

$$B = \nu \mathbf{1} + g B' , \quad V = g V' , \quad \nu, g \in \mathbb{C} . \quad (3.52)$$

We have not found any generalization of the spectral flow to the Lagrangian systems with higher-order P - or Q -structures (*i.e.* nonlinear \mathcal{F} or $\widetilde{\mathcal{F}}$ functions).

3.2.4 Consistent truncations off shell

Reality conditions can be imposed off shell by requiring the action to be either real or purely imaginary, *viz.*

$$(S_{\text{bulk}}^{\text{cl}})^\dagger = \epsilon_S S_{\text{bulk}}^{\text{cl}} , \quad (3.53)$$

leading to the following reality conditions on the Lagrange multipliers and the function $\widetilde{\mathcal{F}}$ appearing in the generalized P -structure:

$$(U_{[p]})^\dagger = \epsilon_S (\epsilon_{\mathbb{R}})^{n+\frac{p}{2}} U_{[p]} , \quad (V_{[p]})^\dagger = -\epsilon_S (\epsilon_{\mathbb{R}})^{n+\frac{p+1}{2}} V_{[p]} , \quad (3.54)$$

$$(\widetilde{\mathcal{F}}_0(\lambda))^\dagger = -\epsilon_{\mathbb{R}} \widetilde{\mathcal{F}}_0(\epsilon_S (\epsilon_{\mathbb{R}})^n \lambda^\dagger) , \quad \left(\widetilde{\mathcal{F}}_I(\lambda) \right)^\dagger = \widetilde{\mathcal{F}}_{\bar{I}}(\epsilon_S (\epsilon_{\mathbb{R}})^n \lambda^\dagger) , \quad (3.55)$$

$$\left(\widetilde{\mathcal{F}}_{I\bar{J}}(\lambda) \right)^\dagger = \epsilon_{\mathbb{R}} \widetilde{\mathcal{F}}_{J\bar{I}}(\epsilon_S (\epsilon_{\mathbb{R}})^n \lambda^\dagger) . \quad (3.56)$$

From $\text{Tr}[\pi_k(\cdot)] = \text{Tr}[\cdot]$ it follows that in the case of π_k -projection then the k -parities must be correlated as follows:

$$\pi_k(U_{[p]}) = -\epsilon_k^{n+\frac{p}{2}} U_{[p]} , \quad \pi_k(V_{[p]}) = \epsilon_k^{n+\frac{p+1}{2}} V_{[p]} , \quad (3.57)$$

$$\widetilde{\mathcal{F}}_0(-(\epsilon_k)^n \lambda) = \epsilon_k \widetilde{\mathcal{F}}_0(\lambda) , \quad \widetilde{\mathcal{F}}_I(-(\epsilon_k)^n \lambda) = (-1)^{I+1} \widetilde{\mathcal{F}}_I(\lambda) , \quad (3.58)$$

$$\widetilde{\mathcal{F}}_{I\bar{J}}(-(\epsilon_k)^n \lambda) = \epsilon_k (-1)^{I+\bar{J}} \widetilde{\mathcal{F}}_{I\bar{J}}(\lambda) . \quad (3.59)$$

To fix spacetime parity one may impose $(\epsilon, \tilde{\epsilon} = \pm 1)$

$$P(U_{[p]}) = \epsilon(\epsilon\tilde{\epsilon})^{n+\frac{p}{2}} U_{[p]} , \quad P(V_{[p]}) = (\epsilon\tilde{\epsilon})^{n+\frac{p+1}{2}} V_{[p]} , \quad (3.60)$$

$$\widetilde{\mathcal{F}}_0(\epsilon(\epsilon\tilde{\epsilon})^n \lambda) = \epsilon\tilde{\epsilon} \widetilde{\mathcal{F}}_0(\lambda) , \quad \widetilde{\mathcal{F}}_{\bar{I}}(\lambda) = \widetilde{\mathcal{F}}_I(\epsilon(\epsilon\tilde{\epsilon})^n \lambda) , \quad \widetilde{\mathcal{F}}_{I\bar{J}}(\lambda) = \epsilon\tilde{\epsilon} \widetilde{\mathcal{F}}_{J\bar{I}}(\epsilon(\epsilon\tilde{\epsilon})^n \lambda) . \quad (3.61)$$

Finally, assuming $\text{Tr}[\tau(\cdot)] = \text{Tr}[\cdot]$, the projection to the minimal bosonic model takes the form

$$\tau(U_{[p]}) = (-1)^{n+\frac{p}{2}} U_{[p]} , \quad \tau(V_{[p]}) = (-1)^{n+\frac{p-1}{2}} V_{[p]} , \quad (3.62)$$

$$\widetilde{\mathcal{F}}_0((-1)^n \lambda) = \widetilde{\mathcal{F}}_0(\lambda) , \quad \widetilde{\mathcal{F}}_I((-1)^n \lambda) = \widetilde{\mathcal{F}}_I(\lambda) , \quad (3.63)$$

$$\widetilde{\mathcal{F}}_{I\bar{J}}((-1)^n \lambda) = \widetilde{\mathcal{F}}_{I\bar{J}}(\lambda) . \quad (3.64)$$

3.3 Even-dimensional bulk ($p \in 2\mathbb{N} + 1$)

In the case of an even-dimensional bulk, say of dimension $\hat{p} + 1 = 2n$, one has the action

$$S_{\text{bulk}}^{\text{cl}}[A, B; S, T] = \int_M \text{Tr} \left[S \star DB + T \star (F + \mathcal{F}) + \mathcal{W}(S; J^I, J^{\bar{I}}, J^{I\bar{J}}) \star T \right] , \quad (3.65)$$

where \mathcal{W} is an interaction \star -function obeying

$$\mathcal{W}(-\lambda) = \mathcal{W}(\lambda) , \quad \mathcal{W}(0) = 0 , \quad (3.66)$$

and the form degrees are assigned as follows:

$$A = \sum_{m=1,3,\dots,\hat{p}} A_{[m]} , \quad B = \sum_{m=0,2,\dots,\hat{p}-1} B_{[m]} , \quad (3.67)$$

$$S = \sum_{m=1,3,\dots,\hat{p}} S_{[m]} , \quad T = \sum_{m=0,2,\dots,\hat{p}-1} T_{[m]} . \quad (3.68)$$

The variational principle yields the generalized curvatures

$$\mathcal{R}^A = F + \mathcal{U} + \mathcal{W}(S) , \quad \mathcal{R}^B = DB - (T\partial_S) \star \mathcal{W}(S) , \quad (3.69)$$

$$\mathcal{R}^S = DS + (T\partial_B) \star \mathcal{F} , \quad \mathcal{R}^T = DT + [S, B]_\star . \quad (3.70)$$

The action is gauge invariant and the equations of motion are integrable in the case of

$$\text{bilinear } Q\text{-structure} : \mathcal{F} = J \star B , \quad (3.71)$$

for which the integrability of \mathcal{R}^T follows using the identity

$$\{S, (T\partial_S) \star \mathcal{W}\}_\star \equiv [T, \mathcal{W}]_\star , \quad (3.72)$$

that holds for general even \star -functions \mathcal{W} . The Cartan gauge transformations off shell are given by the on-shell transformations.

4 Discussions

Let us summarize our results, speculate on future directions and finally conclude by trying to place our work and ideas into the more general state of affairs.

4.1 Summary

In this paper we presented an action principle for a duality extended version of Vasiliev's equations for interacting higher spin gauge fields (including gravity) in four dimensions.

The duality extended version consists of differential forms of degrees $p \in \{0, 1, 2, \dots\}$ forming two master fields $B = B_{[0]} + B_{[2]} + \dots$ and $A = A_{[1]} + A_{[3]} + \dots$, and their Lagrange multipliers which are differential forms of dual form degrees of degrees $\hat{p} - p$ where $\hat{p} + 1$ is the dimension of the base manifold (including the twistor Z -space). The initial and boundary data associated with the Lagrange multipliers are removed by means of boundary conditions compatible with the variational principle. As a result, the Lagrange multipliers can be set equal to zero on shell, leaving A and B subject to the unfolded equations of motion $dA + A \star A + J \star B \approx 0$ and $dB + A \star B - B \star A \approx 0$ where $J = J_{[2]} + J_{[4]}$ is a

closed and central element. This system contains Vasiliev's original equations in degrees zero and one, viz. $dA_{[1]} + A_{[1]} \star A_{[1]} + J_{[2]} \star B_{[0]} \approx 0$ and $dB_{[0]} + A_{[1]} \star B_{[0]} - B_{[0]} \star A_{[1]} \approx 0$.

An important point that remains to be established is whether the coupling $J_{[4]} \star B$ is nontrivial in the sense that it cannot be redefined away. In Vasiliev's original system, the coupling $J_{[2]}^2 \star B$ (and its hermitian conjugate) is nontrivial; it is indeed this term that reproduces the nontrivial interactions in the second order in curvature in the effective unfolded equations of motion in the perturbative expansion around a non-degenerate vierbein [40]. The reason $J_{[2]}^2 \star B$ is nontrivial is that the central term $J_{[2]}^2$ contains the inner Kleinian κ (that becomes a Dirac delta function in the Weyl order of the (Y, Z) -oscillator algebra). We note that also $J_{[4]}$ contains such “singular” elements, namely $J_{[4]}^{1\bar{2}} \star B$ (and its hermitian conjugate) and $J_{[4]}^{\bar{1}2} \star B$.

The duality-extended (A, B) -system is perturbatively equivalent to Vasiliev's original $(A_{[1]}, B_{[0]})$ -system:

- i) both systems share the same Weyl zero-form $B_{[0]}$; this master field contains the initial data associated with the Weyl curvature tensors, which contain one-particle states and other local deformations of the system such as for example the massive parameters of the black-hole solution of [41].
- ii) the master fields with positive form degree (including $A_{[1]}$) bring gauge functions on shell. In topologically broken phases, the boundary values of gauge functions associated with topologically broken gauge symmetries may contribute to observables; see Appendix A. Thus the original and duality-extended systems share the same observable gauge functions in the unbroken phase (where no gauge functions are observable) and in broken phases where projections of $A_{[1]}$ are broken (such as for example the π -odd projection containing the ordinary vierbein).

We wish to stress, however, that if one has an exact solution to the duality-extended (A, B) -system, then it by construction contains an exact solution to the original system. As known from [42], there exist exact solutions of the original system for which the connections exhibit critical behaviors for finite amplitudes of $B_{[0]}$ (as can be described invariantly using zero-form invariants). Thus it is not clear whether a given exact solution to the original system can be uplifted to the duality-extended system, as new critical phenomena may arise in potentials in the duality-extended sector.

We also wish to stress that the action principle involves an integration over a base manifold given by the product of an ordinary commuting base manifold (containing four-dimensional spacetime) and the non-commutative twistor Z -space. The Lagrangian also contains an additional integration over the internal twistor Y -space — which one may think of as contracting indices related to various representations of an internal higher-spin Lie algebra.

In this sense, if one was to take our action principle seriously as a starting point for quantizing higher-spin gravity, one would have to address the issue of boundary conditions on the internal connection $(A_\alpha, A_{\dot{\alpha}})$ in Z -space. In the standard perturbative expansion in the Weyl zero-form $B_{[0]}$, it is usually assumed that $(A_\alpha, A_{\dot{\alpha}})$ is pure gauge in the limit where $B_{[0]}$ vanishes. However, as found in [42], there are “topologically nontrivial” exact solutions based on projectors in which $(A_\alpha, A_{\dot{\alpha}})$ remains nontrivial for vanishing $B_{[0]}$, whose physical meaning remains to be understood better.

4.2 Outlook: AKSZ-BV quantum action and unfolded quantum field theory

The action principle proposed in this paper is an example of a generalized Hamiltonian action principle for an associative free differential algebra on a noncommutative base manifold. More generally, as far as the off-shell formulation of free differential algebras is concerned, one may think of three different levels of complexity depending on whether the algebra is associative and commutative, or associative and non-commutative, or of strongly homotopy associative type. In the commutative case, the BV quantum action is of the AKSZ-BV type and it has been proposed that the perturbative quantization (with suitable boundary conditions on Lagrange multipliers) yields master theories of the homotopy type (with ℓ -ary products arising via terms in the Hamiltonian that are of ℓ -th order in the Lagrange multipliers).

In our case, there exists a quantum action of AKSZ-BV type which we shall present elsewhere. Moreover, the classical $(A, B; U, V)$ system extends naturally to the strongly homotopy associative case and there are indications that its completion off shell leads to an AKSZ-BV-like quantum action (within a suitable Noether procedure). It is thus tempting to speculate that there exist quantum theories based on layers on “ n -quantized” unfolded quantum field theories such that each layer is the master theory of the layer below with radiative corrections interpreted as a topological sum, giving rise to third-quantization.

Pursuing these ideas, one is led to attempt to identify Vasiliev’s equations as the master equations for an underlying first-quantized topological open string: the system on the commutative manifold appears related to an underlying A-model; and the system on the noncommutative twistor space appears related to a B-model [43]. More generally, one may deform the bulk action with various topological vertex operators inserted on finite-dimensional sub-manifolds: these are gauge-invariant functionals whose variations vanish on shell (so that the standard first-order action is an example of such a deformation) and whose values on shell can be interpreted as amplitudes [11, 17, 18]. There are many such deformations, each of which one may seek to relate to an underlying first-quantized dual, such as for example the holographic dual in three dimensions for which one may propose a topological vertex operator that is a four-form [44].

The perturbations of the bulk action by various operators also provides a systematic approach to symmetry breaking mechanisms: for example, one has topological mechanisms (homotopy phases),

spontaneous mechanisms (classical solutions) and dynamical mechanisms (radiative corrections).

More radically, one may go so far as to elevate the aforementioned layered structure of unfolded quantum field theories into a *quantum gauge principle*, *i.e.* a set of mathematical rules that are nontrivial in the sense that they are meant to hold for any *physical* (quantum) system. In particular, the Cartan integrable free differential algebra of the n th layer, with its exterior derivative d (on a base manifold) and Q -structure (in a target space), should arise from the BRST operator of the $(n - 1)$ -quantized system (subject to radiative corrections but with trivial topology as the topological sum of the $(n - 1)$ st layer should correspond to the radiative corrections of the n th layer). In other words, the quantum gauge principle is meant to contain Cartan's version of Weyl's classical gauge principle.

In other words, the idea is that generic quantum system should *not* abide by the quantum gauge principle making it nontrivial. We believe, however, that the Vasiliev system is a candidate for (a massless sector of) a system compatible with the quantum gauge principle.

4.3 Conclusions

As far as four-dimensional higher-spin gravities are concerned, the only fully nonlinear models that are known up to this date are those that have been obtained within Vasiliev's formalism. Vasiliev's formalism provides a general framework for higher-spin gravities based on free differential algebras on noncommutative manifolds taking their values in internal associative (super)algebras.

All models arising within this framework are based on one and the same universal equation of motion; different models arise by choosing different base manifolds and associative algebras. In this sense, all models arising within Vasiliev's framework can be viewed as various Yang–Mills and supersymmetric extensions of a basic minimal bosonic model consisting perturbatively of a scalar field, a metric and a tower of Fronsdal tensors of ranks $\{4, 6, \dots\}$.

Strictly speaking, these perturbative formulations arise only under a set of extra assumptions (on boundary conditions in twistor spaces); whether the resulting perturbative models exhaust all mathematical possibilities within the perturbative Fronsdal programme is an open problem though there are uniqueness theorems to low orders.

Remarkably, notwithstanding its somewhat peculiar features in comparison to the more traditional approach to lower-spin gravities, the perturbative expansions of Vasiliev's equations around its anti-de Sitter vacuum appear paradigmatic as far as holography is concerned, that is, it reproduces the simplest possible candidates for holographic duals of higher-spin gravities [45, 46, 47, 48]; see for example the recent works in [49, 50, 51, 52].

Vasiliev's equations admit, however, exact solutions that involve moduli that are not visible in the perturbative Fronsdal Programme (for example solutions activating the internal connections in twistor

space but not the Weyl tensors). The formalism also admits extensions by differential forms whose exterior derivatives vanish identically in the linearized approximation which one may think of as analogs of the three-dimensional gauge fields².

Taken altogether, the state of affairs motivates a more careful examination of whether the full field content of Vasiliev's unfolded formalism should be treated as the actual fundamental field content. In Vasiliev's system, Fronsdal's equations appear in a precise perturbative sector and most likely the complete theory requires to consider the twistor Z -space on an equal footing with spacetime. In this approach, the aim becomes to include all unfolded variables (differential forms) into the action principle, which leads more or less directly to the type of generalized Hamiltonian bulk actions considered in this paper and in fact already considered in [3], albeit in its simpler version without any Poisson structure.

These action principles lend themselves naturally to the BRST treatment leading to generalized AKSZ-BV models, which is the stage at which we are now. The resulting open problem is how to connect back to the perturbative quantization scheme within the Fronsdal Programme with its clear physical interpretation. To this end it is natural to examine various perturbations of the bulk action, which we leave for future studies.

Note added: The results in this paper were partly presented by P.S. at the IVth International Sakharov Conference on Physics, 18-23 May 2009, Lebedev Institute (Moscow), and at the International workshop on Gauge Theories, Supersymmetry, and Mathematical Physics, 6-10 April 2010, Lyon, France.

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A Free differential algebras on non-commutative base manifolds

Vasiliev's on-shell formulation of higher-spin gravity makes use of a version of unfolded dynamics that is based on associative free differential algebras with central and closed terms. Such an algebra encodes the following key structures:

$$(\mathfrak{B}, \mathfrak{A}, \star, d; \mathfrak{J}; \mathcal{I}, \overrightarrow{\mathcal{D}}; t) ,$$

²These forms appear in the k -independent part of $B_{[0]}$ and the k -linear part of $A_{[1]}$.

and it describes the moduli space $\mathcal{M}_{\mathfrak{t}}$ of \mathfrak{A} -valued sections $\{Z^i\}_{i \in \mathcal{I}}$ over a noncommutative base manifold \mathfrak{B} , subject to universally Cartan integrable flatness conditions on generalized curvatures

$$\mathcal{R}^i := dZ^i + \mathcal{Q}^i(Z, J) \approx 0, \quad i \in \mathcal{I}, \quad (\text{A.1})$$

and defined modulo unbroken Cartan gauge transformations generated by \mathfrak{t} , a subalgebra of the Cartan gauge algebra \mathfrak{g} .

The $\{Z^i\}$ are the fundamental (classical) fields of the unfolded system; we refer to Z^i as the master field of flavor i . The master fields are differential forms in degrees $p_i \equiv \deg(Z^i) \in \mathbb{N}$ (including zero-forms). They can be acted upon with the exterior derivative d and composed using the associative noncommutative product $\star \equiv \star \wedge$ combining the product on \mathfrak{A} and the composition of differential forms on \mathfrak{B} (represented by symbols). The following rules apply:

$$\deg(Z^i \star Z^j) = \deg(Z^i) + \deg(Z^j), \quad \deg(d) = 1, \quad (\text{A.2})$$

$$d(Z^i \star Z^j) - (dZ^i) \star Z^j - (-1)^{\deg(Z^i)} Z^i \star (dZ^j) \equiv 0, \quad (\text{A.3})$$

$$(Z^i \star Z^j) \star Z^k - Z^i \star (Z^j \star Z^k) \equiv 0. \quad (\text{A.4})$$

Locally, in the coordinate charts $\mathfrak{B}_\xi \subset \mathfrak{B}$, labelled here by an additional chart index, the sections have local representatives

$$Z_\xi^i \in \Omega(\mathfrak{B}_\xi) \otimes \mathfrak{A}. \quad (\text{A.5})$$

The structure functions $\mathcal{Q}^i(Z, J)$ in (A.1) are given by \star -power expansions in Z^i and an additional set $\{J^I\}$ of globally defined elements that are central and closed, viz.

$$J^I \in \Omega(\mathfrak{B}) \otimes \mathfrak{A}, \quad dJ^I \equiv 0, \quad J^I \star Z^i - Z^i \star J^I \equiv 0, \quad (\text{A.6})$$

hence generating a closed and central subalgebra

$$\mathfrak{J} \subset \Omega(\mathfrak{B}) \otimes \mathfrak{A}. \quad (\text{A.7})$$

The structure functions can thus be presented as

$$\mathcal{Q}^i = \sum_n \mathcal{Q}_{j_1, \dots, j_n}^i(J^I) \star Z^{j_1} \star \dots \star Z^{j_n} \quad (\text{A.8})$$

with coefficients $\mathcal{Q}_{j_1, \dots, j_n}^i(J^I) \in \mathfrak{J}$ that need not be graded symmetric in their lower flavor indices (due to the non-commutativity of \star). The universal Cartan integrability of (A.1) is tantamount to compatibility with $d^2 \equiv 0$ on base manifolds \mathfrak{B} of arbitrary dimension. Using the notation for \star -vector fields (see Appendix C), this amounts to that

$$\overrightarrow{\mathcal{Q}} \star \overrightarrow{\mathcal{Q}} \equiv 0, \quad \overrightarrow{\mathcal{Q}} := \mathcal{Q}^i(Z^j, J^I) \partial_i, \quad (\text{A.9})$$

or equivalently, that the coefficients obey

$$\sum_{n_1+n_2=n-1} \sum_{m=1}^{n_1} \mathcal{Q}_{j_1, \dots, j_{m-1}, k, j_m, \dots, j_{n_1-1}}^i(J^I) \star \mathcal{Q}_{j_{n_1}, \dots, j_n}^k(J^I) \equiv 0, \quad (\text{A.10})$$

where the flavor indices j_1, \dots, j_n are not subject to any graded symmetry.

The universal Cartan integrability implies that the constraint surface remains invariant under the Cartan gauge transformations

$$\delta_\varepsilon Z^i \equiv \overrightarrow{\mathcal{F}}_{\varepsilon, d\varepsilon; Z} \star Z^i := d\varepsilon^i - \overrightarrow{\varepsilon} \star \mathcal{Q}^i, \quad \overrightarrow{\varepsilon} := \varepsilon^i \partial_i, \quad (\text{A.11})$$

which are linear in gauge parameters ε^i and in general nonlinear in Z^i . These transformations form a soft gauge algebra \mathfrak{g} that exponentiates into generalized (or soft) group elements

$$\mathcal{G}_{\lambda, Z} := \exp_\star \overrightarrow{\mathcal{F}}_{\lambda, d\lambda; Z} \quad (\text{A.12})$$

generated by (finite) gauge functions λ^i . The space \mathcal{M}_ξ of locally defined solutions to $\mathcal{R}_\xi^i := dZ_\xi^i + \mathcal{Q}^i(Z_\xi^j, J^I) \approx 0$ is given formally by Cartan gauge orbits, *viz.*

$$\mathcal{M}_\xi = \{ \mathcal{G}_{\lambda, Z} \star Z^i : \lambda = \lambda_\xi, Z^i = Z_{C_\xi}^i \}, \quad (\text{A.13})$$

where λ_ξ^i and $Z_{C_\xi}^i$ are locally-defined gauge functions and reference solutions, respectively; the reference solution obeys i) the constraints $dZ_{C_\xi}^i + \mathcal{Q}^i(Z_{C_\xi}, J) \approx 0$; ii) an initial datum $(Z_{C_\xi}^i|_{[0]})|_{p_\xi} = C_\xi^i$ where $p_\xi \in \mathfrak{B}_\xi$ is a base point and $(\cdot)|_{[0]}$ denotes the projection to zero form degree; and iii) a physical gauge condition (to select a well-defined particular solution and avoid over-representation). Interestingly enough, the unfolded formulation of higher-spin gravities appears amenable to the implementation of the above form of Cartan integrability at least in sub-sectors of the theory.

The moduli space $\mathcal{M}_\mathfrak{l}$ is obtained by first gluing together locally-defined modules \mathcal{M}_ξ by means of transition functions valued in the unbroken gauge algebra $\mathfrak{l} \subseteq \mathfrak{g}$, *viz.*

$$\mathcal{M}_\xi \cong \mathcal{G}_\xi^{\xi'} \star \mathcal{M}_{\xi'}, \quad \mathcal{G}_\xi^{\xi'} := \exp_\star \overrightarrow{\mathcal{F}}_{t_\xi^{\xi'}, dt_\xi^{\xi'}; Z_{\xi'}^i}, \quad t_\xi^{\xi'} \in \mathfrak{l}, \quad (\text{A.14})$$

where the parameters are defined on (cylinders homotopic to) the overlaps $\mathfrak{B}_\xi \cap \mathfrak{B}_{\xi'}$ (we are assuming that $\mathfrak{B} = \bigcup_\xi \mathfrak{B}_\xi$). The gluing compatibility implies that

$$Z_{C_\xi}^i = Z_C^i \quad \text{for all } \xi, \quad (\text{A.15})$$

where thus C is (gauge non-invariant) constant of motion, and that

$$\mathcal{G}_\xi^{\xi'} = \mathcal{G}_\xi \star (\mathcal{G}_{\xi'})^{-1}, \quad (\text{A.16})$$

which is a nontrivial gluing condition on the gauge functions. The coordinates on $\mathcal{M}_\mathfrak{l}$ are gauge-invariant and intrinsically defined observables $\mathcal{O}_\mathfrak{l}$, that is, functionals of the master fields constructed out of local

functionals that are manifestly \mathfrak{l} -invariant off shell and intrinsically defined, *i.e.* independent under any particular choices of local data on the base manifold and hence manifestly diffeomorphism invariant (consequently non-local). The manifest \mathfrak{l} -invariance implies that $\mathcal{G}_\xi^{\xi'} \sim \mathcal{U}_\xi \star \mathcal{G}_\xi^{\xi'} \star (\mathcal{U}_{\xi'})^{-1}$ where \mathcal{U}_ξ is generated by \mathfrak{l} . Thus, in view of (A.16), one has that

$$\mathcal{G}_\xi \sim \mathcal{U}_\xi \star \mathcal{G}_\xi \quad \text{where } \mathcal{U}_\xi \text{ is generated by } \mathfrak{l}, \quad (\text{A.17})$$

that is, the gauge functions in $\mathcal{M}_\mathfrak{l}$ can be taken to be valued in the coset $\mathfrak{g}/\mathfrak{l}$.

For example, one may consider homotopy charges given by integrals

$$\mathcal{O} := \oint_{\Sigma'} (\omega^R + k^R), \quad \Sigma' \in [\Sigma] \quad (\text{A.18})$$

over nontrivial p_R -cycles $[\Sigma]$ of p_R -forms $\omega^R[Z, J]$ and $k^R[Z, J]$ that are manifestly \mathfrak{l} -invariant, *i.e.*

$$\delta_\varepsilon(\omega^R, k^R) \equiv 0, \quad \varepsilon \in \mathfrak{l}, \quad (\text{A.19})$$

and defined by the equivariant cohomology system

$$d\omega^R + f^R(\omega) \approx 0, \quad f^R(\omega)|_{\Sigma_{\text{cyl}}} \approx dk^R|_{\Sigma_{\text{cyl}}}, \quad (\text{A.20})$$

where Σ_{cyl} is a cylinder of finite thickness containing Σ ; the homotopy invariance of de Rham cohomology classes then implies that $H^{p_R+1}(\Sigma_{\text{cyl}}) = 0$ so that $f^R|_{\Sigma_{\text{cyl}}}$ must be exact, that is, given by the exterior derivative of some p_R -form k^R that is globally defined on Σ (and hence gauge invariant). Thus the integral over Σ , which must necessarily be split into several charts, say $\{\Sigma_\xi\}$, makes sense and is independent of the choice of Σ' . A variation $\delta_\varepsilon \lambda^i = \varepsilon^i$ in the gauge functions thus induces a change in $(\omega^R + k^R)|_{\Sigma_\xi}$ given by

$$\delta_\varepsilon(\omega^R + k^R)|_{\Sigma_\xi} = dX_\xi(\varepsilon_\xi), \quad (\text{A.21})$$

where $X_\xi(\varepsilon_\xi)$ is a linear functional in ε_ξ^i . By the \mathfrak{l} -invariance, one has that $X_\xi(\varepsilon_\xi)$ is invariant under \mathfrak{l} -transformations that act simultaneously on Z^i and the gauge parameter (*c.f.* the BRST treatment where the gauge parameter is promoted into a ghost). It follows that

$$\delta_\varepsilon \mathcal{O}_\mathfrak{l} = \sum_\xi \oint_{\partial \Sigma_\xi} X_\xi(\varepsilon_\xi), \quad (\text{A.22})$$

which can be split into contributions from chart boundaries in the interior of \mathfrak{B} and from true boundaries of \mathfrak{B} . The former must cancel identically if one assumes that the choice of where to cut the interior of \mathfrak{B} into charts should not be of no importance. Taking into account the signs coming from orientation, this is a consequence of the fact that $\{\lambda^i\}$ forms a globally defined section (of the soft \mathfrak{l} -bundle) as stated in (A.16). One thus has

$$\delta_\varepsilon \mathcal{O}_\mathfrak{l} = \sum_\xi \oint_{\partial \mathfrak{B} \cap \partial \Sigma_\xi} X_\xi(\varepsilon_\xi), \quad (\text{A.23})$$

that is the only physical dependence of the gauge functions enters via their boundary values, which one may view as an unfolded version of the holographic principle.

B Duality extension

We consider an associative free differential algebra consisting of master fields Z^i and structure coefficients $\mathcal{Q}_{j_1, \dots, j_n}^i(J^I)$ of fixed degrees, say $\deg(Z^i) = p_i \in \mathbb{N}$ and $\deg(\mathcal{Q}_{j_1, \dots, j_n}^i) = p_{j_1 \dots j_n}^i \in 2\mathbb{N}$. This system can always be duality extended (without adding any new local degrees of freedom) by i) replacing Z^i by $\widehat{Z}^i := \sum_k Z_{[p_i+2k]}^i$; and ii) exploiting field redefinitions to introduce coupling constants $g_{[0]}$ and then replace these by $\widehat{g}(J^I) := \sum_k g_{[2k]}^I$. It follows that the extended system $\{\widehat{Z}^i, \widehat{g}\}$ contains the original system $\{Z_{[p_i]}^i, g_{[0]}\}$ as a consistent subsystem, though the added master fields $Z_{[p_i+2k]}^i$ with $k > 0$ cannot in general be set equal to zero, since they are sourced from $\{Z_{[p_i]}^i\}$ via terms involving the new couplings $g_{[2k]}^I$ with $k > 0$.

One may refer to the duality extension as non-trivial if the central elements cannot be removed by redefining the master fields; we are not aware of any general condition that guarantees non-triviality.

C Further details: \star -vector fields and Cartan integrability

In this Appendix we go into the technical details of \star -functions, \star -vector fields and Cartan integrability that were introduced in Appendix A. Let us first recall the general idea of a free differential algebra on a non-commutative base manifold \mathfrak{B} consisting of graded associative algebras \mathfrak{R}_ξ generated by sets $\{Z_\xi^i\}_{i \in \mathcal{I}}$ of locally-defined differential forms subject to generalized curvature constraints

$$\mathcal{R}_\xi^i := dZ_\xi^i + \mathcal{Q}^i(Z_\xi, J) \approx 0, \quad (\text{C.1})$$

where $\overrightarrow{\mathcal{Q}} := \mathcal{Q}^i \partial_i$ is a composite \star -vector field of total degree one subject to the Cartan integrability condition

$$\overrightarrow{\mathcal{Q}} \star \mathcal{Q}^i \equiv 0. \quad (\text{C.2})$$

Here we use the following notation and conventions:

- (i) ξ labels charts $\mathfrak{B}_\xi \subset \mathfrak{B}$ with coordinates Ξ_ξ^M of degree zero and differentials $d\Xi_\xi^M$ of degree one generating \mathbb{N} -graded associative \star -product algebras

$$\Omega_\xi \equiv \text{Env}[\Xi_\xi^I, d\Xi_\xi^I] \quad (\text{C.3})$$

modulo the graded \star -commutators

$$[\Xi_\xi^M, \Xi_\xi^N]_\star = 2i\Pi^{MN}, \quad [\Xi_\xi^M, d\Xi_\xi^N]_\star = 0, \quad [d\Xi_\xi^M, d\Xi_\xi^N]_\star = 0, \quad (\text{C.4})$$

where Π^{MN} is a constant matrix (defining a canonical Poisson structure $\Pi = \Pi^{MN} \partial_M \otimes \partial_N$);

(ii) the action of the exterior derivative $d = d\Xi_\xi^M \partial / \partial \Xi_\xi^M$ in Ω_ξ is defined by declaring that

$$d(\Xi_\xi^M) = d\Xi_\xi^M, \quad d(f \star g) = (df) \star g + (-1)^{\deg f} f \star (dg), \quad (C.5)$$

for elements $f, g \in \Omega$ such that f has fixed form degree $\deg f$; one has

$$d^2 \equiv 0. \quad (C.6)$$

(iii) the locally-defined differential forms $Z_\xi^i \in \Omega_\xi^{[p_i]} \otimes \Theta^i$, where $\Omega_\xi^{[p_i]}$ is the subspace of Ω_ξ of fixed form degree m_i and Θ^i can be either are finite-dimensional internal tensors (such as for example Lorentz tensors) or sectors of an internal associative algebra \mathfrak{A} ;

(iv) the graded associative \star -product algebra $\mathfrak{R}_\xi := \text{Env}[Z_\xi^i] \otimes \mathfrak{J}$ where \mathfrak{J} is a space of central and d -closed elements (including the identity), *i.e.* if $\mathcal{F}(Z_\xi^i) \in \mathfrak{R}_\xi$ then

$$\mathcal{F} = \sum_{n \geq 0} \mathcal{F}_{j_1 \dots j_n} \star Z_\xi^{j_1} \star \dots \star Z_\xi^{j_n}, \quad \mathcal{F}_{j_1 \dots j_n} \in \mathfrak{J}; \quad (C.7)$$

(v) a composite \star -vector field $\overrightarrow{\mathcal{X}}$ is a graded inner derivation of \mathfrak{R} , *i.e.* if $\mathcal{F}, \mathcal{F}' \in \mathfrak{R}$ then

$$\overrightarrow{\mathcal{X}} \star (\mathcal{F} \star \mathcal{F}') = (\overrightarrow{\mathcal{X}} \star \mathcal{F}) \star \mathcal{F}' + (-1)^{\deg(\overrightarrow{\mathcal{X}})\deg(\mathcal{F})} \mathcal{F} \star (\overrightarrow{\mathcal{X}} \star \mathcal{F}'), \quad (C.8)$$

provided that $\overrightarrow{\mathcal{X}}$ and \mathcal{F} have fixed degrees. In components, one writes $\overrightarrow{\mathcal{X}} := \mathcal{X}^i(Z^j)\partial_i$ where $\mathcal{X}^i := \mathcal{X} \star Z^i$ (and $\partial_i \equiv \overrightarrow{\partial}_i$). The graded bracket between two composite \star -vector fields is defined by

$$[\overrightarrow{\mathcal{X}}, \overrightarrow{\mathcal{X}'}]_\star \star \mathcal{F} := \overrightarrow{\mathcal{X}} \star (\overrightarrow{\mathcal{X}'} \star \mathcal{F}) - (-1)^{\deg(\overrightarrow{\mathcal{X}})\deg(\overrightarrow{\mathcal{X}'})} \overrightarrow{\mathcal{X}'} \star (\overrightarrow{\mathcal{X}} \star \mathcal{F}), \quad (C.9)$$

is a degree-preserving graded Lie bracket, *i.e.* $[\overrightarrow{\mathcal{X}}, \overrightarrow{\mathcal{X}'}]_\star$ is a graded inner derivation obeying the graded Jacobi identity $[[\overrightarrow{\mathcal{X}}, \overrightarrow{\mathcal{X}'}]_\star, \overrightarrow{\mathcal{X}''}]_\star + \text{graded cyclic} \equiv 0$. In components, one has

$$[\overrightarrow{\mathcal{X}}, \overrightarrow{\mathcal{X}'}]_\star = \left(\overrightarrow{\mathcal{X}} \star \mathcal{X}'^i - (-1)^{\deg(\overrightarrow{\mathcal{X}})\deg(\overrightarrow{\mathcal{X}'})} \overrightarrow{\mathcal{X}'} \star \mathcal{X}^i \right) \partial_i. \quad (C.10)$$

The Cartan integrability condition (C.2), that can be rewritten $[\overrightarrow{\mathcal{Q}}, \overrightarrow{\mathcal{Q}}]_\star \equiv 0$, amounts to that $\overrightarrow{\mathcal{Q}}$ is a nilpotent composite \star -vector field of degree one. This condition ensures that the generalized curvature constraints $\mathcal{R}^i \approx 0$ are compatible with $d^2 \equiv 0$ without further algebraic constraints on the generating elements Z_ξ^i . One can also show that the nilpotency of $\overrightarrow{\mathcal{Q}}$ is separately equivalent to that the generalized curvatures \mathcal{R}^i obey the generalized Bianchi identities

$$d\mathcal{R}^i - \overrightarrow{\mathcal{R}} \star \mathcal{Q}^i \equiv 0, \quad \text{where} \quad \overrightarrow{\mathcal{R}} := \mathcal{R}^i \partial_i, \quad (C.11)$$

and transform into each other under the following Cartan gauge transformations

$$\delta_\varepsilon Z^i \equiv \mathcal{T}_\varepsilon^i := d\varepsilon^i - \vec{\varepsilon} \star \mathcal{Q}^i, \quad \text{where} \quad \vec{\varepsilon} := \varepsilon^i \partial_i \quad (\text{C.12})$$

and where ε^i is an element in $\Omega \otimes \Theta^i$ that is considered infinitesimal and independent of Z^i , viz.

$$\delta_\varepsilon \mathcal{R}^i = -\vec{\mathcal{R}} \star ((\vec{\varepsilon} \star \mathcal{Q}^i)). \quad (\text{C.13})$$

The closure relation reads

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] Z^i = \delta_{\varepsilon_{12}} Z^i - \vec{\mathcal{R}} \star \varepsilon_{12}^i, \quad (\text{C.14})$$

where the combined parameters ε_{12}^i 's are given by

$$\varepsilon_{12}^i = -\frac{1}{2} [\vec{\varepsilon}_1, \vec{\varepsilon}_2]_\star \star \mathcal{Q}^i. \quad (\text{C.15})$$

The above results can easily be obtained upon introducing the even \star -vector field

$$\vec{\mathcal{V}}_\varepsilon := (\vec{\varepsilon} \star \mathcal{Q}^i) \partial_i \quad (\text{C.16})$$

and using the following set of identities which are consequences of the first one:

$$[\vec{\mathcal{Q}}, \vec{\mathcal{Q}}]_\star \equiv 0, \quad [\vec{\mathcal{Q}}, \vec{\mathcal{V}}_\varepsilon]_\star \equiv 0, \quad [\vec{\mathcal{V}}_{\varepsilon_1}, \vec{\mathcal{V}}_{\varepsilon_2}]_\star \equiv [\vec{\mathcal{Q}}, \vec{\varepsilon}_{12}]_\star, \quad (\text{C.17})$$

where we recall the all the commutators are graded-commutators.

As discussed above, the local representatives \mathfrak{R}_ξ are glued together on overlaps $\mathfrak{B}_\xi \cup \mathfrak{B}_{\xi'}$ by means of the transitions $Z_\xi^i = \mathcal{G}_\xi^{\xi'} \star Z_{\xi'}^i$, where the transition functions $\mathcal{G}_\xi^{\xi'}$ are soft group elements given by \star -exponentials of the Cartan gauge transformations as in (A.12). From the Leibnitz' rule (C.8) it follows that these transitions are indeed isomorphisms, viz.

$$\mathcal{G} \star \mathcal{F}(Z) = \mathcal{F}(\mathcal{G} \star Z), \quad \mathcal{G} \star (\mathcal{F} \star \mathcal{F}') = (\mathcal{G} \star \mathcal{F}) \star (\mathcal{G} \star \mathcal{F}'). \quad (\text{C.18})$$

We would like to show that, if Z^i satisfies the star-product equation $dZ^i + \mathcal{Q}^i(Z^j) \approx 0$, then $Z_\lambda^i := (\exp_\star[\vec{\mathcal{F}}_{\lambda, Z}]) \star Z^i$ where $\vec{\mathcal{F}}_\lambda := d\lambda^i \partial_i - \vec{\mathcal{V}}_\lambda$ [see (C.16)] satisfies the equation $dZ_\lambda^i + \mathcal{Q}^i(Z_\lambda, J) \approx 0$, thereby exhibiting the fundamental integrability of the unfolded equations in the case where the free differential algebra \mathcal{A} is endowed with a non-commutative star-product. We recall that

Lemma: The following commutation relation is true: $[\vec{\mathcal{F}}_\lambda, d]_\star \approx 0$, where the weak equality means an equality on the surface $\Sigma \equiv \{dZ^i + \mathcal{Q}^i(Z, J)\} = 0$.

Proof of the Lemma: On the surface Σ , the total exterior derivative $d \approx \vec{\mathcal{Q}} - \vec{\Lambda}$, where

$$\vec{\Lambda} := d\lambda^i \frac{\partial}{\partial \lambda^i}. \quad (\text{C.19})$$

The proof is tantamount to showing that $[\vec{\mathcal{F}}_\lambda, \vec{\mathcal{Q}} - \vec{\Lambda}]_\star \star Z^i = 0 = [\vec{\mathcal{F}}_\lambda, \vec{\mathcal{Q}} - \vec{\Lambda}]_\star \star \lambda^i$ because then, using the fact that $[\vec{\mathcal{F}}_\lambda, \vec{\mathcal{Q}} - \vec{\Lambda}]_\star$ is a \star -vector field, it follows that $[\vec{\mathcal{F}}_\lambda, \vec{\mathcal{Q}} - \vec{\Lambda}]_\star \star \mathcal{F}(Z, \lambda) = 0$ for an arbitrary star-product function $\mathcal{F}(Z, \lambda)$.

- (a) First of all, it is trivial to see that $[\vec{\mathcal{F}}_\lambda, \vec{\mathcal{Q}} - \vec{\Lambda}]_\star \star \lambda^i = 0$. Indeed, it gives $\vec{\mathcal{F}}_\lambda(d\lambda^i)$ which vanishes³.
- (b) That $[\vec{\mathcal{F}}_\lambda, \vec{\mathcal{Q}} - \vec{\Lambda}]_\star \star Z^i = 0$ is more difficult to show. For that, we write

$$\mathcal{Q}^i = \sum_n \mathcal{Q}_{j_1 \dots j_n}^i(J) \star Z^{j_1} \star \dots \star Z^{j_n}$$

where $\mathcal{Q}_{j_1 \dots j_n}^i \in \mathfrak{J}$ and compute

$$\begin{aligned} \vec{\mathcal{Q}} \star (\mathcal{T}_\lambda \star Z^i) &= - \sum_n \sum_{\beta < \alpha=1}^n (-1)^{j_{\beta+1} + \dots + j_{\alpha-1}} \mathcal{Q}_{j_1 \dots j_n}^i \star Z^{j_1} \star \dots \star \mathcal{Q}^{j_\beta} \star \dots \star \lambda^{j_\alpha} \star \dots \star Z^{j_n} \\ &\quad - \sum_n \sum_{\alpha < \beta=1}^n (-1)^{1+j_\alpha + \dots + j_\beta} \mathcal{Q}_{j_1 \dots j_n}^i \star Z^{j_1} \star \dots \star \lambda^{j_i} \star \dots \star \mathcal{Q}^{j_\beta} \star \dots \star Z^{j_n} \quad , \\ \vec{\mathcal{F}}_\lambda \star (\mathcal{Q} \star Z^i) &= [d\lambda^k - (\lambda^j \partial_j) \star \mathcal{Q}^k] \partial_k \star \mathcal{Q}^i \quad , \\ \vec{\Lambda} \star (\mathcal{T}_\lambda \star Z^i) &= - \sum_n \sum_{\alpha=1}^n \mathcal{Q}_{j_1 \dots j_n}^i \star Z^{j_1} \star \dots \star d\lambda^{j_\alpha} \star \dots \star Z^{j_n} \quad , \quad \mathcal{T}_\lambda \star (\Lambda \star Z^i) = 0 \quad . \end{aligned}$$

Regrouping all the terms, we find

$$[\vec{\mathcal{F}}_\lambda, \vec{\mathcal{Q}} - \vec{\Lambda}]_\star \star Z^i = \mathcal{Q}^j \partial_j \star [(\lambda^k \partial_k) \star \mathcal{Q}^i] - [(\lambda^k \partial_k) \star \mathcal{Q}^j] \partial_j \star \mathcal{Q}^i \quad (\text{C.20})$$

which vanishes identically due to the second identity of (C.17).

Therefore, since $[\vec{\mathcal{F}}_\lambda, \vec{\mathcal{Q}} - \vec{\Lambda}]_\star$ is a star-product vector field, it follows that $[\vec{\mathcal{F}}_\lambda, \vec{\mathcal{Q}} - \vec{\Lambda}]_\star \star \mathcal{F}(Z, \lambda) = 0$ for an arbitrary star-product function $\mathcal{F}(Z, \lambda)$. \square

Using the above Lemma, we have that $Z_\lambda^i := (\exp_\star[\vec{\mathcal{F}}_\lambda]) \star Z^i$ satisfies the equation $dZ_\lambda^i + \mathcal{Q}^i(Z_\lambda^j) \approx 0$, since $dZ_\lambda^i \equiv d[(\exp_\star[\vec{\mathcal{F}}_\lambda]) \star Z^i] = (\exp_\star[\vec{\mathcal{F}}_\lambda]) \star dZ^i \approx -(\exp_\star[\vec{\mathcal{F}}_\lambda]) \star \mathcal{Q}^i(Z) \equiv -\mathcal{Q}^i(Z_\lambda)$. This proves the formal Cartan integrability of the star-product unfolded equations.

D The Vasiliev equations

In the case of Vasiliev's equations, the master fields are locally-defined operators of the form

$$O_\xi(X_\xi^M, P_M^\xi, dX_\xi^M, dP_M^\xi; Z^\alpha, dZ^\alpha, Y^\alpha; e^i) \quad , \quad (\text{D.1})$$

where the non-vanishing commutators among the coordinates are

$$[X^M, P_N]_\star = i\delta_N^M \quad , \quad [Y^\alpha, Y^\beta]_\star = 2iC^{\alpha\beta} \quad , \quad [Z^\alpha, Z^\beta]_\star = -2iC^{\alpha\beta} \quad , \quad (\text{D.2})$$

³We consider the algebra where the fields $\{Z^i\}$ and $\{\lambda^i\}$ are considered as independent, in accordance with the *BRST* treatment of gauge systems.

with charge conjugation matrix⁴ $C^{\alpha\beta} = \epsilon^{\alpha\beta}$ and $C^{\dot{\alpha}\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}}$ and where $\{e^i\}$, $i = 1, 2$, are two outer Kleinian operators. The operators are represented by symbols $f[O_\xi]$ obtained by going to specific bases for the operator algebra which one may also think of as ordering prescriptions⁵. One may think of the symbols as functions $f(X, P, Z; Y; dX, dP, dZ)$ (with variables composed using commutative juxtaposition) on a correspondence space

$$\mathfrak{C} = \bigcup_{\xi} \mathfrak{C}_\xi, \quad \mathfrak{C}_\xi = \mathfrak{B}_\xi \times \mathfrak{Y}, \quad \mathfrak{B}_\xi = \mathfrak{M}_\xi \times \mathfrak{Z} \quad (\text{D.3})$$

equipped with a suitable associative star-product operation \star which reproduces, in the space of symbols, the composition rule for operators. Working within a restricted class of orderings, referred to as universal orderings, the exterior derivative on \mathfrak{B} is given by

$$d = dX^M \partial_M + dP_M \partial^M + q, \quad q := dZ^\alpha \partial_\alpha. \quad (\text{D.4})$$

The master fields of the (duality-unextended) minimal bosonic model are an adjoint one-form

$$A = W + V, \quad (\text{D.5})$$

$$W = dX^M W_M(X, P, Z; Y) + dP_M W^M(X, P, Z; Y), \quad V = dZ^\alpha V_\alpha(X, P, Z; Y), \quad (\text{D.6})$$

and a twisted-adjoint zero-form

$$\Phi = \Phi(X, P, Z; Y); \quad (\text{D.7})$$

these fields obey the following projection and reality conditions⁶:

$$\tau(A, \Phi) = (-A, \pi(\Phi)), \quad (A, \Phi)^\dagger = (-A, \pi(\Phi)), \quad (\text{D.8})$$

⁴We raise and lower quartet and doublet indices using the conventions $\Lambda^\alpha = C^{\alpha\beta} \Lambda_\beta$, and $\lambda^\alpha = \epsilon^{\alpha\beta} \lambda_\beta$ and $\lambda_\alpha = \lambda^\beta \epsilon_{\beta\alpha}$, and we use the notation $\Lambda \cdot \Lambda' = \Lambda^\alpha \Lambda'_\alpha$, $\lambda \cdot \lambda' = \lambda^\alpha \lambda'_\alpha$ and $\bar{\lambda} \cdot \bar{\lambda}' = \bar{\lambda}^{\dot{\alpha}} \bar{\lambda}'_{\dot{\alpha}}$.

⁵The symbols are thus defined modulo similarity transformations generated by inner automorphisms (related to the higher-spin gauge transformations) as well as changes of the order prescription, that is, changes of basis of the operator algebra. These types of transformations may have a drastic effect on the mathematical nature of the symbols, that may change from being a smooth or real analytic into being singular or even distributions. Thus, in order to extract physically meaningful information from the master fields, one needs to develop the notion of observables \mathcal{O} , namely functionals of the locally-defined master fields that are invariant under both gauge transformations and re-orderings. The construction of such functionals introduces various geometric concepts into the theory, such as flat connections, covariantly constant sections (going into decorated Wilson loops), equivariantly closed forms (used to define homotopy charges) and metrics (that yield minimal areas of closed cycles).

⁶Here we are focusing on the models containing spacetimes with Lorentzian signature and negative cosmological constant; for other signatures and signs of the cosmological constant, see [42].

where the maps $\tau, \pi, \bar{\pi}$ and \dagger are defined by $d \circ (\tau, \pi, \bar{\pi}, \dagger) = (\tau, \pi, \bar{\pi}, \dagger) \circ d$ and⁷

$$\pi(y_\alpha, \bar{y}_{\dot{\alpha}}; z_\alpha, \bar{z}_{\dot{\alpha}}) = (-y_\alpha, \bar{y}_{\dot{\alpha}}; -z_\alpha, \bar{z}_{\dot{\alpha}}), \quad \pi(f \star g) = \pi(f) \star \pi(g), \quad (\text{D.9})$$

$$\bar{\pi}(y_\alpha, \bar{y}_{\dot{\alpha}}; z_\alpha, \bar{z}_{\dot{\alpha}}) = (y_\alpha, -\bar{y}_{\dot{\alpha}}; z_\alpha, -\bar{z}_{\dot{\alpha}}), \quad \bar{\pi}(f \star g) = \bar{\pi}(f) \star \bar{\pi}(g), \quad (\text{D.10})$$

$$\tau(y_\alpha, \bar{y}_{\dot{\alpha}}; z_\alpha, \bar{z}_{\dot{\alpha}}) = (iy_\alpha, i\bar{y}_{\dot{\alpha}}; -iz_\alpha, -i\bar{z}_{\dot{\alpha}}), \quad \tau(f \star g) = (-1)^{fg} \tau(g) \star \tau(f), \quad (\text{D.11})$$

$$(y_\alpha, \bar{y}_{\dot{\alpha}}; z_\alpha, \bar{z}_{\dot{\alpha}})^\dagger = (\bar{y}_{\dot{\alpha}}, y_\alpha; \bar{z}_{\dot{\alpha}}, z_\alpha), \quad (f \star g)^\dagger = (-1)^{fg} g^\dagger \star f^\dagger. \quad (\text{D.12})$$

The τ -projection removes all terms that are associated with the unfolded description of spacetime fermions as well as spacetime bosons with odd spin.

The full equations of motion for the minimal bosonic model with the simplest interaction freedom amount to the statement that the full curvature $F = dA + A \star A$ is proportional to Φ , viz $F + \Phi \star J = 0$, via a deformed symplectic two-form J that is defined globally on correspondence space and obeying $\tau(J) = -J = J^\dagger$ and

$$dJ = 0, \quad [J, f]_\pi = 0, \quad (\text{D.13})$$

for any f obeying $\pi \bar{\pi}(f) = f$, and where we have defined $[f, g]_\pi = f \star g - g \star \pi(f)$. In the minimal model,

$$J = -\frac{i}{4}(b dz^2 \kappa + \bar{b} d\bar{z}^2 \bar{\kappa}), \quad (\text{D.14})$$

where the chiral Klein operators are given in the normal-ordering by

$$\kappa = \exp(iy^\alpha z_\alpha), \quad \bar{\kappa} = \kappa^\dagger = \exp(-i\bar{y}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}}). \quad (\text{D.15})$$

The quantities κ and $\bar{\kappa}$ are the Klein operators of the chiral Heisenberg algebras generated by (y_α, z_α) and $(\bar{y}_{\dot{\alpha}}, \bar{z}_{\dot{\alpha}})$. The two-dimensional complexified Heisenberg algebra $[u, v]_\star = 1$ has the Klein operator $\kappa = \cos_\star(\pi v \star u)$, which anti-commutes with u and v and squares to 1. Hence κ remains invariant under the canonical $SL(2; \mathbb{C})$ -symmetry that becomes manifest in Weyl order, where the symbol of κ is thus proportional to the two-dimensional Dirac delta function. It follows that $(\kappa, \bar{\kappa})$ is invariant under $SL(4; \mathbb{C}) \times \overline{SL}(4; \mathbb{C})$, which is broken by dz^2 and $d\bar{z}^2$ down to a global $GL(2; \mathbb{C}) \times \overline{GL}(2; \mathbb{C})$ symmetry of the Vasiliev system, generated by diagonal $SL(2; \mathbb{C}) \times \overline{SL}(2; \mathbb{C})$ transformations and the exchange $(y_\alpha, z_\alpha) \leftrightarrow (iz_\alpha, -iy_\alpha)$. The latter symmetry is hidden in the formulation in terms of differentials on Z -space while it becomes manifest in the deformed-oscillator formulation.

By making use of field redefinitions $\Phi \rightarrow \lambda F$ with $\lambda \in \mathbb{R}$, $\lambda \neq 0$, the parameter b in J can be taken to obey

$$|b| = 1, \quad \arg(b) \in [0, \pi]. \quad (\text{D.16})$$

⁷The rule $(f \star g)^\dagger = g^\dagger \star f^\dagger$ holds for both real and chiral integration domain.

The phase breaks parity except in the following two cases:

$$\text{Type A model (parity-even physical scalar)} : b = 1 , \quad (\text{D.17})$$

$$\text{Type B model (parity-odd physical scalar)} : b = i . \quad (\text{D.18})$$

The integrability of $F + \Phi \star J = 0$ implies that $D\Phi \star J = 0$, that is, $D\Phi = 0$, where the twisted-adjoint covariant derivative $D\Phi = d\Phi + A \star \Phi - \Phi \star \pi(A)$. This constraints is integrable since

$$D^2\Phi = F \star \Phi - F \star \pi(\Phi) = -\Phi \star J \star \Phi + \Phi \star \pi(\Phi) \star J = 0 , \quad (\text{D.19})$$

using the constraint on F and (D.13).

Thus, in summary, the unfolded system describing the minimal higher-spin gravity with simplest possible interaction term, is given by⁸

$$F + \Phi \star J = 0 , \quad D\Phi = 0 , \quad dJ = 0 , \quad (\text{D.20})$$

$$F = dA + A \star A , \quad D\Phi = d\Phi + [A, \Phi]_\pi , \quad (\text{D.21})$$

and the kinematic constraints D.8 which imply $[A, J]_\pi = 0 = [\Phi, J]_\pi$. The integrability is manifest in as much as the associativity of the \star -product in manifest. The integrability implies the Cartan gauge transformations⁹

$$\delta_\epsilon A = D\epsilon , \quad \delta_\epsilon \Phi = -[\epsilon, \Phi]_\pi , \quad (\text{D.22})$$

for zero-form gauge parameters $\epsilon(X, P, Z; Y)$ obeying the same kinematic constraints as the master one-form, *i.e.* $\tau(\epsilon) = -\epsilon$ and $(\epsilon)^\dagger = -\epsilon$. The closure of the gauge transformations reads

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{\epsilon_{12}} , \quad \epsilon_{12} = [\epsilon_1, \epsilon_2]_\star , \quad (\text{D.23})$$

defining the algebra $\mathfrak{hs}(4)$.

The symbols of the Kleinians are distributions on the doubled twistor space whose precise form depend on the choice of ordering scheme (that can thus be adapted to different physical problems); for example, in overall Weyl order they localize to Dirac delta functions (that are useful in trace calculations) while in overall normal order they become Gaussians (that are useful in perturbation theory).

The singular nature of the Kleinians implies that the source term $\Phi \star J$ cannot be absorbed into a field redefinition [36]. Moreover, upon projection of the full equations to a Lagrangian sub-manifold of the universal phase space, say $P_M = 0$, which can be obtained in an expansion in the zero-form, the

⁸The format applies also to Yang–Mills extended or supersymmetric models; for example, see [53, 54, 55].

⁹These transformations are the canonical transformations of the \star -product algebra generated by (D.2) containing the diffeomorphisms of Lagrangian submanifolds of the unfold.

twistor-space source term induces nontrivial albeit perturbatively defined deformations of the generalized curvatures $dA + A \star A$ and $D\Phi = d\Phi + A \star \Phi - \Phi \star \pi(A)$ of the $\mathfrak{hs}(4)$ -valued connection $A = A|_{Z=P=0}$ and the twisted-adjoint zero-form $\Phi = \Phi|_{Z=P=0}$. Upon further weak-field expansion around large spin-two gauge fields, *i.e.* vierbein $e^{\alpha\dot{\alpha}}$ and Lorentz connection $(\omega^{\alpha\beta}, \bar{\omega}^{\dot{\alpha}\dot{\beta}})$, the deformations contain the canonical linearized source terms for unfolded Fronsdal tensors in accordance with Vasiliev's central on-shell theorem.

In other words, the Vasiliev system contains a set of nontrivial equations of motion for perturbatively defined Fronsdal tensors. The full system contains, however, various other moduli that have either problematic or no description in terms of Fronsdal fields, such as classical solutions with degenerate vierbeins and topological degrees of freedom contained in the internal connection $A_{\underline{\alpha}}$ [42].

Over and above their formal Cartan integrability, the Vasiliev equations exhibit the following more powerful integrable structures:

- The Maurer-Cartan integrability facilitates the explicit construction of solutions using gauge functions [56, 42, 57, 58, 39] and the formal construction gauge-invariant observables [44];
- The zero-forms $S_{\alpha} := z_{\alpha} - 2iA_{\alpha}$ and $S_{\dot{\alpha}} := \bar{z}_{\dot{\alpha}} - 2iA_{\dot{\alpha}}$ the following generalization of Wigner's deformed oscillator algebra with local anyonic deformation parameter Φ , *viz.*

$$\begin{aligned} [S_{\alpha}, S_{\beta}]_{\star} &= -2i\epsilon_{\alpha\beta}(1 - \Phi \star \kappa) \quad , \quad [S_{\dot{\alpha}}, S_{\dot{\beta}}]_{\star} = -2i\epsilon_{\dot{\alpha}\dot{\beta}}(1 - \Phi \star \bar{\kappa}) \quad , \\ [S_{\alpha}, S_{\dot{\beta}}]_{\star} &= 0 \quad , \quad S_{\alpha} \star \Phi + \Phi \star \pi(S_{\alpha}) = 0 \quad , \quad S_{\dot{\alpha}} \star \Phi + \Phi \star \bar{\pi}(S_{\dot{\alpha}}) = 0 \quad , \end{aligned} \quad (\text{D.24})$$

which one may also think of as describing the deformation of the symplectic structure on a submanifold of complex dimension two of the doubled twistor space (of complex dimension four).

These properties have been used to construct classical solutions in [59, 60, 42, 41, 39], for perturbative calculations of the twistor-space vertices $P(W; \Phi)$ and $J(W, W; \Phi)$ in [54] and direct verification of the conjectured correspondence between Vasiliev's four-dimensional higher-spin gravities and three-dimensional conformal field theories [47, 48], first in [61] at the level of cubic scalar self-couplings, and recently for the complete cubics in [49, 50].

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